

1.3e Solving Equations Numerically (Using the Computer)

Many equations cannot be solved algebraically. To solve them we must resort to approximate methods involving graphs and computers. Here is the process:

Step 1 - Prepare a function corresponding to the equation:

Assume that the equation's unknown is called x . Use algebra to move everything over to one side of the equation. This puts the equation into the form $0=f(x)$, where $f(x)$ denotes an expression involving x . Then replace the 0 on the LHS by the variable y . The result is $y=f(x)$, ie. y is some function of x , which can now be graphed.

Step 2 - Graph the function and estimate the roots:

Make a graph of the function $y=f(x)$, plotting y vertically versus x horizontally. Find the points where the function crosses the x axis. These are the values of x for which $y=0$ or $f(x)=0$. In other words, these are the required roots of the equation. (Recall the relationship between the quadratic *function* and the quadratic *equation* discussed in Section 1.3b.) Thus from the graph we can find the number of roots and their approximate values. René Descartes, for whom the Cartesian coordinate system is named, discovered this method.

Example 1.30: Use the two steps described above to find the number of solutions and their approximate values for the following equation.

$$x = \frac{1}{x} + \frac{1}{x-5} \quad (1.7)$$

Solution: First bring everything to one side of the equation. There are many ways that this can be done. One way is to simply subtract x from both sides of the equation to get

$$0 = \frac{1}{x} + \frac{1}{x-5} - x.$$

Then replace the 0 on the LHS by y .

$$y = \frac{1}{x} + \frac{1}{x-5} - x \quad (1.8)$$

This function is graphed in Fig. 1.4(a).

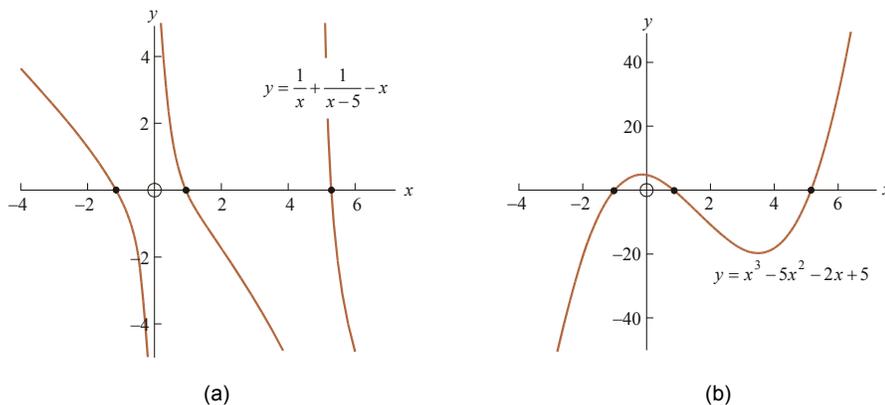


Figure 1.4 There are many ways to carry out step 1. (a) and (b) show two ways. The graphs are different but they agree on the number and location of the roots (the x axis crossings).

The roots are the points where $y=0$. The graph shows that there are three such points:

$$x \approx -1, x \approx 1 \text{ and } x \approx 5.$$

Another way to prepare the function is by first clearing denominators in Eq. (1.7) by multiplying through by $x(x-5)$ and then bringing everything to one side:

$$x = \frac{1}{x} + \frac{1}{x-5}$$

$$\leftrightarrow x^2(x-5) = (x-5) + (x)$$

$$\leftrightarrow x^3 - 5x^2 - 2x + 5 = 0.$$

Now graph the function

$$y = x^3 - 5x^2 - 2x + 5. \tag{1.9}$$

This *cubic* function, shown in Fig. 1.4(b), looks completely different from the previous function but crosses the x axis at exactly the same points. Thus we find the same three roots for the equation: $x \approx -1, x \approx 1$ and $x \approx 5$. Which function is better, (1.8) or (1.9)? Probably (1.9) since its graph is continuous while that of (1.8) is not.



Step 3 - Use a computer to locate the x intercepts with precision:

There are several computer methods to do this. Here we will look at the **bisection method**. In book 2 we study another method called the Newton-Raphson method which is based on calculus. Whatever the method, the roots are found one at a time.

The bisection method

The method begins with an interval that is known to *bracket* (contain) the root to be found. For example in Fig. 1.5 the root is bracketed by the interval $[L_1, R_1]$ (i.e. by L_1 on the left and R_1 on the right). Notice that y is positive at L_1 and negative at R_1 . Since the sign of y is different at the two endpoints y must equal zero somewhere between these endpoints (assuming that the curve is continuous). The bisection method is based on this idea. The method is *iterative*, meaning that we repeat some procedure over and over. That procedure is to **bisect** the interval, evaluate y at the interval's midpoint and look at its sign (+ or -). The midpoint replaces the endpoint that has the same sign of y as it does, thus becoming the new endpoint.

Figure 1.5 We wish to solve the equation $f(x)=0$. The solution (root) is the big dot where the corresponding curve $y=f(x)$ crosses the x axis. The bisection method starts with an interval $[L_1, R_1]$ that is known to contain the root. It bisects the interval and replaces it with a new interval $[L_2, R_2]$ that is either the left half or the right half of the old interval. The half is chosen so that the root remains inside the new interval.

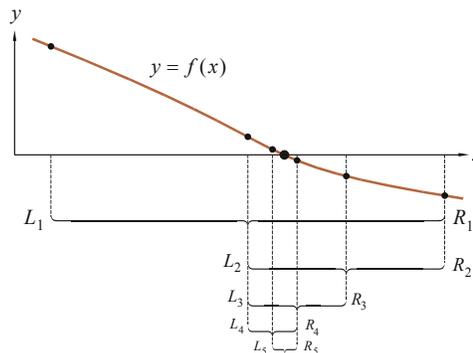


Fig. 1.5 shows several iterations for a hypothetical function. L_n and R_n are the left and right endpoints of the interval at the n^{th} iteration. With each iteration the width of the interval is halved. When the width is small enough the procedure is stopped.

The box below contains Visual Basic code that implements the Bisection Method. It can be used in an Excel spreadsheet. Do the following:

- Start Excel. At the top are the tabs “File, Home, Insert, ...” Look for the Developer tab. If it is not there then click on File>Options>Customize ribbon and check the Developer checkbox on the right side of the dialog box.
- On the Developer tab click the Visual Basic button. This opens Excel’s Visual Basic Editor in a new window.
- In the Editor click on Insert>Module to open a module into which the code can be typed.
- Type (or copy) all of the code in the box below into the module. You can now close the Visual Basic Editor window and go back to Excel.
- Type the following formula into any cell in an Excel spreadsheet:

=Bisection (0, 2, 0.001)

This searches for a root in the interval between 0 and 2 to a precision of 0.001. A message box displays once each iteration showing the progress. (This message box is for diagnostic purposes only and the line of code that displays it can be deleted or commented out by putting an apostrophe in front of it.) Finally the found root value 0.886 is displayed in the cell. Similarly, using the formula =Bisection (-2, 0, 0.001) finds the root -1.085 and the formula =Bisection (4, 6, 0.001) finds the root 5.200.

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Function Bisection(Left As Double, Right As Double, Accuracy As Double) As Double
  Dim Middle As Double

  Do
    Middle = (Left + Right) / 2
    If SameSign(Y(Middle), Y(Left)) Then Left = Middle Else Right = Middle
    MsgBox (Middle & Chr(177) & Right - Left)           'Optional: comment out or delete this line
  Loop Until Right - Left < Accuracy Or Y(Middle) = 0

  Bisection = Middle
End Function

Function Y(X As Double) As Double                               'The function whose root is to be found.
  Y = X ^ 3 - 5 * X ^ 2 - 2 * X + 5
End Function

Function SameSign(A As Double, B As Double) As Boolean
  'Returns True if A and B are both positive or both negative; returns False otherwise.
  If (A > 0 And B > 0) Or (A < 0 And B < 0) Then SameSign = True Else SameSign = False
End Function

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Details of the code: There are three functions. The main function, *Bisection*, implements the Bisection Method. The other two functions are $Y(X)$, which defines the function whose root is to be found, and $SameSign(A, B)$ which simply returns True or False depending on whether or not two numbers A and B have the same sign. In function *Bisection* the variables *Left*, *Middle* and *Right* are the x coordinates of the left endpoint, midpoint and right endpoint of the interval in the present iteration. *Accuracy* is the interval width at which the program is to stop. Defining the variables as Double means that 16 sig. figs. of accuracy are used in the course of the calculation.