10.6 Applications and Extensions of Fourier Series

Filtering of a Waveform by a Parallel *RLC* circuit

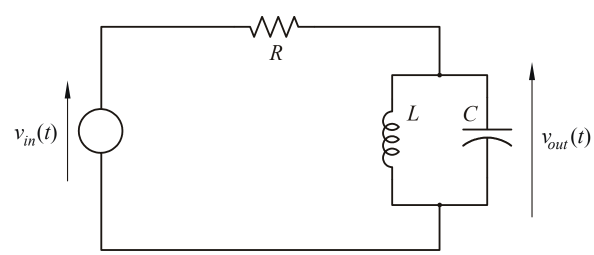

Figure 10.32 Suppose that we apply a waveform v_{in} and wish to find v_{out}. This can be done by expressing v_{in} as a Fourier series, finding v_{out} for each term of the Fourier series, and adding. We will see that the inductor tends to short out low frequencies and the capacitor tends to short out high frequencies.

Consider the circuit shown in Fig. 10.32. Suppose that we apply a given $v_{in}(t)$ and wish to calculate $v_{out}(t)$. First suppose that the input voltage is a simple *sinusoidal* waveform with angular velocity ω, say $v_{in}(t) = A\sin(\omega t)$. In this case we learned previously how to use complex numbers to find $v_{out}(t)$. The calculation consists of these steps.

Step 1. Calculate the complex impedance of the parallel *LC* part of the circuit.

$$Z_{parallel} = \frac{(j\omega L)\left(-j\dfrac{1}{\omega C}\right)}{(j\omega L)+\left(-j\dfrac{1}{\omega C}\right)} = \frac{\dfrac{L}{C}}{j\left(\omega L - \dfrac{1}{\omega C}\right)}$$

Step 2. Apply the voltage divider law.

$$\frac{v_{out}}{v_{in}} = \frac{\dfrac{\dfrac{L}{C}}{j\left(\omega L - \dfrac{1}{\omega C}\right)}}{\dfrac{\dfrac{L}{C}}{j\left(\omega L - \dfrac{1}{\omega C}\right)} + R} = \underbrace{\frac{\dfrac{L}{C}}{\dfrac{L}{C} + jR\left(\omega L - \dfrac{1}{\omega C}\right)}}_{\text{a complex number that depends on } \omega} \tag{10.52}$$

Notice that for fixed values of R, L and C the ratio v_{out}/v_{in} is a complex number whose value depends on ω, the angular velocity of v_{in}. For example if $R = 1000\ \Omega$, $L = 0.1$ H, and $C = 1000$ μF, then

$$\frac{v_{out}}{v_{in}} = \frac{1}{1 + \left(\omega - \dfrac{10{,}000}{\omega}\right) j}.$$

If we express v_{out}/v_{in} in polar form, namely $v_{out}/v_{in} = A\angle\varphi$, and plot its magnitude A and its angle φ versus ω we get the graphs shown in Fig. 10.33. Notice that if $\omega = 100$ rads/sec then $A = 1$ and $\varphi = 0$ implying that $v_{out} = v_{in}$ but the farther ω is from 100 rads/sec the closer A is to zero implying that v_{out} is also closer to zero.

Step 3. Cross-multiply (10.52) to express v_{out} in terms of v_{in}.

$$v_{out} = \frac{\dfrac{L}{C}}{\dfrac{L}{C} + jR\left(\omega L - \dfrac{1}{\omega C}\right)} \cdot v_{in} \tag{10.53}$$

Express the input voltage $v_{in}(t) = A\sin(\omega t)$ as the phasor $v_{in} = A\angle 0°$, substitute the values of R, L, C, ω and v_{in} into (10.53) and do the complex arithmetic. The result is v_{out} expressed as a phasor which we can then rewrite as a sinusoidal function of t.

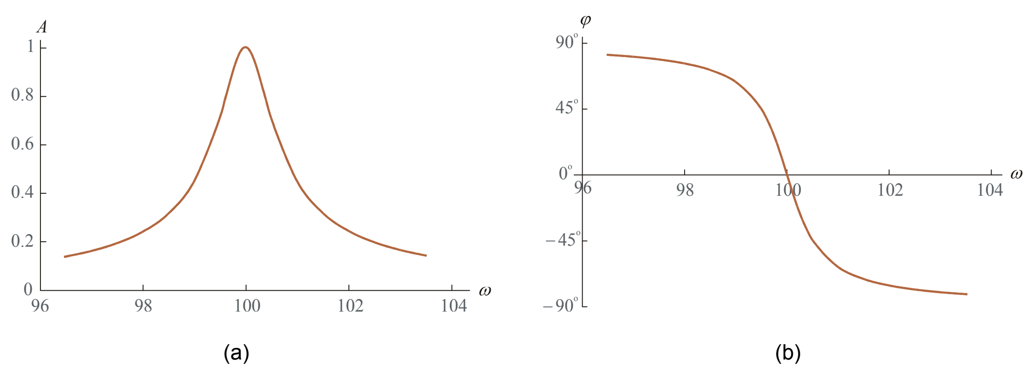

Figure 10.33 The ratio v_{out}/v_{in} is a complex number $A\angle\varphi$ in polar form where A and φ both depend on ω as shown in (a) and (b). When $\omega = 100$ rads/sec then $A = 1$ and $\varphi = 0°$ implying that $v_{out} = v_{in}$ but if ω is much different from 100 then $A \approx 0$ meaning that $v_{out} \approx 0$ also.

Now suppose that the input voltage v_{in} is an *arbitrary waveform*. In that case we can express v_{in} as a Fourier series and apply Eq. (10.53) to each term of the Fourier series to get corresponding term of the Fourier series for v_{out}. Add up all the terms and we have v_{out}. Here is an example.

Example 10.13: In the circuit in Fig. 10.32 suppose that R = 1000 Ω, L = 0.1 H, and C = 1000 μF, and that the input voltage v_{in} is a *square* waveform with period 20π msec and with one cycle given by

$$v_{in}(t) = \begin{cases} 10 \text{ volts}, & 0 < t < 10\pi \text{ msec} \\ 0 \text{ volts}, & 10\pi < t < 20\pi \text{ msec} \end{cases}$$

Find the output voltage.

Solution: For these values of R, L and C, Eq. (10.53) reduces to

$$v_{out} = \frac{1}{1+\left(\omega - \frac{10,000}{\omega}\right)j} \cdot v_{in} \tag{10.54}$$

To get the Fourier series for v_{in} we can scale and translate the result of the square waveform in Example 10.2. First, scale and translate in the vertical direction.

$$f(\theta) = 5 + \frac{20}{\pi}\left\{\frac{\sin\theta}{1} + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \cdots\right\}$$

Next scale in the horizontal direction. Use the fact that the period of our square waveform is $T = 20\pi \times 10^{-3}\,\text{s}$ so the angular velocity of the fundamental is $\omega = \frac{2\pi}{T} = \frac{2\pi}{20\pi \times 10^{-3}\,\text{s}} = 100\,\text{rad/s}$. Make the replacement $\theta = \omega t$ or $\theta = 100t$ and rename $f(\theta) \to v_{in}(t)$. The result is that the Fourier series for our waveform is

$$v_{in}(t) = 5 + \frac{20}{\pi}\sin 100t + \frac{20}{3\pi}\sin 300t + \frac{20}{5\pi}\sin 500t + \cdots$$

Now we consider the terms of v_{in} *one at a time* and calculate the corresponding output v_{out} for each. To get the total v_{out} for our square waveform we then simply add the individual v_{out}'s together. We will demonstrate this for the first three terms of v_{in}.

Term 1: v_{in} = 5. For this term ω = 0 so Eq. (10.54) becomes $v_{out} = \frac{1}{1+\infty} \cdot v_{in} = 0$. (The inductor looks like a short for DC).

Term 2: $v_{in}(t) = \frac{20}{\pi}\sin 100t$, or in phasor form, $v_{in} = \frac{20}{\pi}\angle 0°$. For this term ω = 100 so Eq. (10.54) becomes $v_{out} = \frac{1}{1+0j} \cdot \frac{20}{\pi}\angle 0° = \frac{20}{\pi}\angle 0°$ and so $v_{out}(t) = \frac{20}{\pi}\sin 100t$, which is unchanged from the input.

Term 3: $v_{in}(t) = \frac{20}{3\pi}\sin 300t$, or in phasor form, $v_{in} = \frac{20}{3\pi}\angle 0°$. For this term $\omega = 300$ so Eq. (10.54) becomes

$$v_{out} = \frac{1}{1+\left(300-\frac{100}{3}\right)j}\cdot\left(\frac{20}{3\pi}\angle 0°\right) = \left(0.00375\angle -89.8°\right)\cdot\left(\frac{20}{3\pi}\angle 0°\right) = 0.00796\angle -89.8°$$

and so

$$v_{out}(t) = 0.00796\sin\left(3000t - 89.8°\right) \approx 0\,.$$

This term is so much smaller than term 2 that we can drop it. The same thing goes for all the following terms. Thus the output for our square wave is just a single term,

$$v_{out}(t) = \frac{20}{\pi}\sin 100t\,.$$

We see that this circuit effectively filters out a single sine wave from the input signal. This makes sense because the circuit has resonance angular velocity $\omega = \frac{1}{\sqrt{LC}} = 100$ rad/s at which value the admittance of the parallel part of the circuit is zero. The fundamental or first harmonic component of our square wave just happens to be at this angular velocity.