3.6 The Newton-Raphson Method

The Newton-Raphson method is a numerical method (i.e. a method that uses numbers and a computer, rather than algebra) for solving for the roots of an equation. It uses the derivative of the equation. There are three steps to the method:

Step 1 – Convert the equation into a function:

Let the unknown in the equation be called x. Use algebra to move everything over to one side of the equation. This puts the equation into the form $0=f(x)$, where $f(x)$ denotes some expression involving x. Then replace the 0 on the LHS by the variable y. The result is $y=f(x)$, ie. y is some function of x.

Let's illustrate with the following equation which arises when an AC current is applied to an electric oscillator having a pair of resonance frequencies.

$$x = \frac{1}{x} + \frac{1}{x-5} \tag{3.20}$$

We can bring everything to one side of the equation by simply subtracting x from both sides of the equation to get

$$0 = \frac{1}{x} + \frac{1}{x-5} - x .$$

Then replace the 0 on the LHS by y.

$$y = \frac{1}{x} + \frac{1}{x-5} - x \tag{3.21}$$

This is not the only way to convert the equation into a function. Another way is to first clear denominators in the equation by multiplying through by $x(x-5)$ and *then* bring everything to one side. Then the original equation,

$$x = \frac{1}{x} + \frac{1}{x-5},$$

becomes

$$x^2(x-5) = 1\,(x-5) + 1\,(x),$$

or

$$0 = x^3 - 5x^2 - 2x + 5.$$

Now replace the 0 on the LHS by y and we get

$$y = x^3 - 5x^2 - 2x + 5. \tag{3.22}$$

Either (3.19) or (3.20) can be used but (3.20) is easier to work with because it is a polynomial.

Step 2 – Graph the function and estimate the roots:

Make a graph of the function $y=f(x)$, plotting y vertically versus x horizontally, and find the points where the function crosses the x axis. These are the values of x for which $y=0$ or $f(x)=0$. In other words, these are the required roots of the equation. Thus from the graph we can find the number of roots and their approximate values.

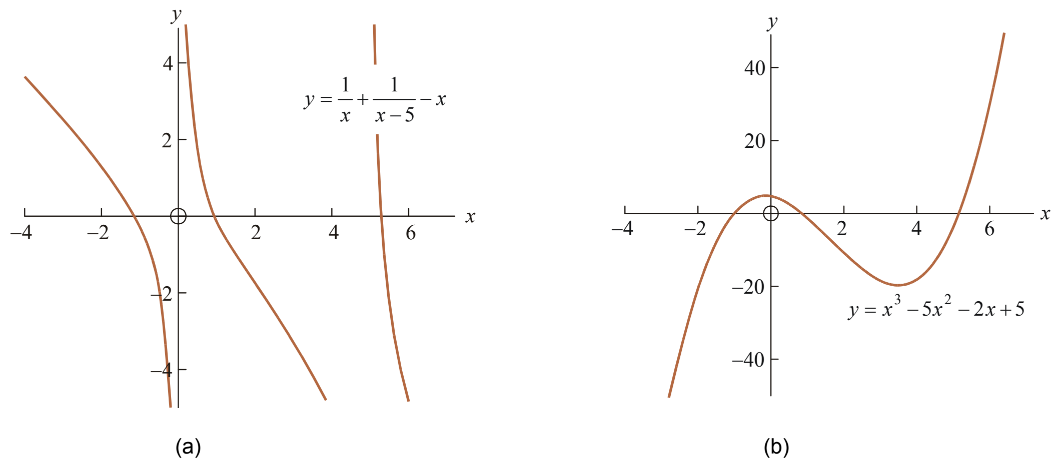

Figure 3.29 The roots of Eq. (3.20) are the *x* intercepts of the functions (3.21) or (3.22). Although the functions look very different they have the same *x* intercepts, $x \approx -1$, $x \approx 1$ and $x \approx 5$.

If we plot (3.21) we get the graph on the left of Fig. 3.29 and if we plot (3.22) we get the graph on the right. Both graphs agree that there are three points where $y = 0$ at approximately

$$x \approx -1, \; x \approx 1 \text{ and } x \approx 5.$$

<u>Step 3 – Use the computer to locate the *x* intercepts with precision</u>:

Now we are ready to go to the computer. There are several computer methods which can be used to find the roots precisely. Whatever the method, the roots are found one at a time. Previously we studied the bisection method. Here we will look at the **Newton-Raphson method**.

The method begins with an initial estimate for the root; call it x_1. Use the height y and the slope dy/dx of the curve evaluated at x_1 to construct a tangent line to the curve at P_1. The point where the tangent line touches the x axis is an improved approximation for the root; call it x_2.

Iterate this procedure (i.e. repeat this process). Use x_2 to get x_3, then use x_3 to get x_4, and so on. Stop when $|x_{i+1} - x_i| < \varepsilon$, where ε is the desired accuracy. In other words, stop when the next estimate of the root is not much different from the present one. Although this iteration can be done by hand, it is an ideal task for a computer because of its repetitive nature.

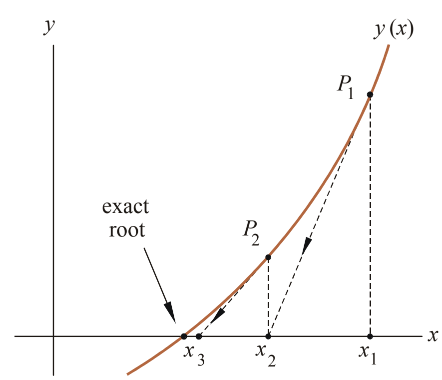

Figure 3.30 The Newton-Raphson method finds the exact root by using an initial estimate x_1 and a tangent line at x_1 to get a better estimate x_2, then uses x_2 to get x_3, and so on.

Formula Connecting x_i and x_{i+1}

We will now derive a formula that connects the next approximation for the root with the present one. Look at Fig. 3.31. The slope of the straight line at P_i is

$$\frac{rise}{run} = \frac{y-0}{x_i - x_{i+1}}.$$

On the other hand the slope of the *curve* at P_i is dy/dx. Since the line is tangent to the curve at P_i we can equate the slopes. This gives

$$\frac{dy}{dx} = \frac{y}{x_i - x_{i+1}}.$$

Solving for x_{i+1} gives us our desired equation.

$$x_{i+1} = x_i - \frac{y}{dy/dx}. \qquad (3.23)$$

This formula gives the improved approximation x_{i+1} in terms of the previous approximation x_i and the height y and slope dy/dx of the curve at x_i.

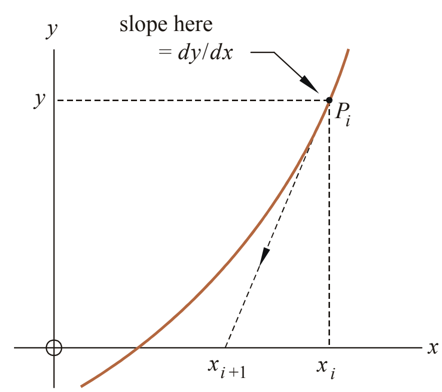

Figure 3.31 Construction used to find x_{i+1}, given x_i. The tangent line has the same slope as the curve at point P_i.

Example 3.18: Use the Newton-Raphson method to find $\sqrt{26}$ accurate to $\pm 10^{-9}$.

Solution: Let $x = \sqrt{26}$ be the equation we wish to solve. The first step is to construct a function corresponding to this equation. We don't want the function to contain the expression $\sqrt{26}$ since this is exactly the quantity whose value we don't know but wish to compute. The easiest thing to do is to square both sides to give $x^2 = 26$. Then subtract 26 to get everything to one side, giving $x^2 - 26 = 0$. Now replace 0 by y. So we wish to find where the function $y = x^2 - 26$ crosses the x axis.

The second step is to graph the function to find how many roots there are, and their approximate locations. In this case this is not necessary since we know that the graph of $y = x^2 - 26$ is a parabola opening upwards and that it crosses the x axis twice; once at $x \approx +5$ and once at $x \approx -5$. We are only interested in the positive root.

The third step is to start with an initial guess for the root and then iterate to find successively better approximations of the root. The iteration formula which gives the next approximation, x_{i+1}, in terms of the present one, x_i, is

$$x_{i+1} = x_i - \frac{y}{dy/dx}.$$

For this problem the function is $y=x^2-26$ and its derivative is $dy/dx=2x$. Both quantities must be evaluated at x_i. Substituting them in gives

$$x_{i+1} = x_i - \frac{x_i^2-26}{2x_i}. \qquad (3.24)$$

The following Excel spreadsheet shows the approximation x_i on the i^{th} iteration. We started with the (terrible) initial guess x_0=10.

	A	B	C	D
1	iteration i	approximation x_i	$y(x_i)$	$y'(x_i)$
2	0	10	74	20
3	1	6.3	13.69	12.6
4	2	5.213492063	1.180499496	10.42698413
5	3	5.100276249	0.012817821	10.2005525
6	4	5.099019668	1.579E-06	10.19803934
7	5	5.099019514	0	10.19803903
8	6	5.099019514	0	10.19803903

The difference between x_5 and x_6 is less than 10^{-9} (we can tell because the display doesn't change) so we can stop with the answer that the square root of 26 is 5.099019514 ± 10^{-9}. Here are the formulas that were entered into the cells:

	A	B	C	D
1	iteration i	approximation x_i	$y(x_i)$	$y'(x_i)$
2	0	10	=B2^2-26	=2*B2
3	1	=B2-C2/D2	=B3^2-26	=2*B3
4	2	=B3-C3/D3	=B4^2-26	=2*B4
5	3	=B4-C4/D4	=B5^2-26	=2*B5
6	4	=B5-C5/D5	=B6^2-26	=2*B6
7	5	=B6-C6/D6	=B7^2-26	=2*B7
8	6	=B7-C7/D7	=B8^2-26	=2*B8

In order for this program to find a root of any other equation, simply change the formulas in columns C and D where the function and its derivative are defined.

Problem Set 3.6 – The Newton-Raphson Method

1. Calculate the square root of 90 to an accuracy of 10^{-9} by using the Newton-Raphson method to find the positive root of the function $y=x^2-90$.

2. Find the three roots of the function $y=x^3-5x^2-2x+5$ to an accuracy of 10^{-9} using the Newton-Raphson method. (Hint: See Fig. 3.29(b). To find all three roots use −1, 1 and 5 as your initial guesses.)