Example 9.19: Implementing the logarithm function on a computer chip.
Some background: Real numbers are stored in a computer in the floating point form, $a \times 2^b$, where a is a number between 1 and 2. (This is similar to scientific notation where real numbers are written in the form $a \times 10^b$, and a is a number between 1 and 10. For example the number 9000 in floating point form is 1.098633×2^{13}).

Arithmetic operations on a computer chip are implemented in a hierarchy. At the very bottom of the hierarchy is addition. Then come subtraction, multiplication and division, which are implemented using modifications of addition (e.g. two's complement for subtraction, left and right shifts and adds for multiplication and division, etc.). Next comes the square root function which is implemented using the Newton-Raphson method and which is basically a sequence of multiplications and additions (see Example 3.18 where calculating $\sqrt{26}$ to 10 sig. figs. took 20 multiplications and 13 additions).

Next come the six elementary functions $\sin x$, $\cos x$, $\tan x$, e^x, $\ln x$, and $\arctan x$. These are implemented by the (careful) use of Maclaurin series. (Note that the square root function can't be expressed as a Maclaurin series. Why?) Finally there are the functions $\arcsin x$, $\arccos x$, $\log x$ and y^x which are implemented using modifications of previous group of six functions.

In this example we will show how the natural logarithm function is implemented on a computer or calculator chip. We begin by noticing that the natural logarithm of a number expressed in floating point form can be written as

$$\ln\left(a \times 2^b\right) = \ln(a) + b\ln(2) = \ln(a) + 0.6931472\,b \tag{9.48}$$

Since a lies between 1 and 2 the problem reduces to that of calculating the logarithm of a number that lies between 1 and 2, and adding to it the product of an integer b multiplied by the constant 0.6931472 which can be stored in ROM. Now look at the Maclaurin series for the natural logarithm function given in Formula #6 in section 9.3, namely

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{9.49}$$

When x is in the range $-0.1 < x < +0.1$ each term of this series is about 10 times smaller than the previous term, so 6 terms will give us about 6 sig. figs. of accuracy. Unfortunately convergence is impractically slow outside this region and the series actually diverges outside the region $-1 < x < +1$. We will now modify this series to produce a new series for the logarithm function that has better convergence. Write two versions of (9.49), first replacing x with y and then replacing x with $-y$:

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \cdots \tag{9.50}$$

$$\ln(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \frac{y^5}{5} - \cdots \tag{9.51}$$

Now subtract one series from the other. Notice that the LHS can be written as a single logarithm and that every second term on the RHS cancels. We get

$$\ln\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{1}{3}y^3 + \frac{1}{5}y^5 + \frac{1}{7}y^7 + \cdots\right). \tag{9.52}$$

If we let $x = \dfrac{1+y}{1-y}$ and keep 6 terms of the series then (9.52) becomes

$$\ln(x) = 2\left(y + \frac{1}{3}y^3 + \frac{1}{5}y^5 + \frac{1}{7}y^7 + \frac{1}{9}y^9 + \frac{1}{11}y^{11}\right) \tag{9.53}$$

where $y = \dfrac{x-1}{x+1}$. This series is a huge improvement over the previous one. We will use it to calculate the logarithm of any number between 1 and 2. This series converges most rapidly when $x = 1$ (and $y = 0$) and least rapidly when $x = 2$ (and $y = \frac{1}{3}$). Even in that case, Eq. (9.53) gives

$$\ln(2) = 2\left(\frac{1}{3} + \frac{1}{81} + \frac{1}{1215} + \frac{1}{15\,309} + \frac{1}{177\,147} + \frac{1}{1\,948\,617}\right) \tag{9.54}$$

Notice that each term is about 10 times smaller than the previous term, so again, keeping 6 terms gives us about 6 sig. figs. of accuracy. We can do one more thing to reduce the number of multiplications in (9.53). We can write the later terms of the series in terms of the earlier terms, like this:

$$\ln(x) = 2y\left(1 + y^2\left(\tfrac{1}{3} + y^2\left(\tfrac{1}{5} + y^2\left(\tfrac{1}{7} + y^2\left(\tfrac{1}{9} + \tfrac{1}{11}y^2\right)\right)\right)\right)\right) \tag{9.55}$$

To evaluate (9.53) requires 36 multiplications and 5 additions. To evaluate (9.55) requires only 17 multiplications and 5 additions. Let's put it all together and compute ln (9000), accurate to 6 sig. figs. First, Eq. (9.48) gives

$$\ln(9000) = \ln\left(1.098633 \times 2^{13}\right) = \ln(1.098633) + 13 \times 0.6931472\,. \tag{9.56}$$

Next, letting $x = 1.098633$ gives $y = \dfrac{x-1}{x+1} = 0.0469986$ so (9.53) gives

$$\ln(1.09863) = 2(0.046999 + 0.000034 + 0.00000005 + \cdots) = 0.0940665$$

Finally, substituting this into (9.56) gives

$$\ln(9000) = 0.0940665 + 13 \times 0.6931472 = 9.10498\,,$$

accurate to six sig. figs.