

# 8. Gauss' and Stokes' Theorems

## 8.1 Flow and an Alternative Definition of Divergence

Given a vector field  $\mathbf{F}$  and a small area element with area  $\Delta A$  and unit normal vector  $\hat{\mathbf{n}}$ , the **flow of  $\mathbf{F}$**  through the area element is defined as

$$\text{flow of } \mathbf{F} \text{ through area element } \Delta A \equiv \mathbf{F} \cdot \hat{\mathbf{n}} \Delta A. \quad (8.1)$$

The name comes from hydrodynamics where  $\rho \mathbf{v}$  is the fluid flux and  $(\rho \mathbf{v}) \cdot \hat{\mathbf{n}} \Delta A$  is the mass per unit time of fluid that flows across the area element.) We will now give an alternate definition of the divergence in terms of the flow. Suppose that  $\Delta V = \Delta x \Delta y \Delta z$  is a small volume element centered on a point  $P = (x, y, z)$  as shown in Fig. 8.1. Then we claim that we can define the divergence of a vector field  $\mathbf{F}$  at  $P$  in terms of the flow like this:

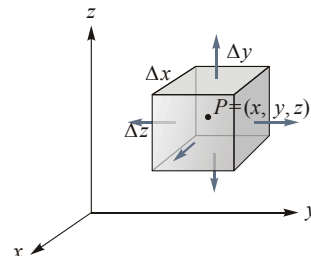
$$\text{divergence of } \mathbf{F} \equiv \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \cdot (\text{total flow of } \mathbf{F} \text{ out through the six faces}) \quad (8.2)$$

### Proof that the new definition agrees with the old definition

Refer to Fig. 8.1. The total flow of  $\mathbf{F}$  out through the six faces of the box is the sum of six terms of the form  $\mathbf{F} \cdot \hat{\mathbf{n}} \Delta \sigma$ ; one coming from each of the front, back, left, right, top and bottom faces, respectively:

$$\begin{aligned} \text{flow} = & \mathbf{F}\left(x + \frac{\Delta x}{2}, y, z\right) \cdot (1, 0, 0) \Delta y \Delta z + \mathbf{F}\left(x - \frac{\Delta x}{2}, y, z\right) \cdot (-1, 0, 0) \Delta y \Delta z + \\ & \mathbf{F}\left(x, y - \frac{\Delta y}{2}, z\right) \cdot (0, -1, 0) \Delta x \Delta z + \mathbf{F}\left(x, y + \frac{\Delta y}{2}, z\right) \cdot (0, 1, 0) \Delta x \Delta z + \\ & \mathbf{F}\left(x, y, z + \frac{\Delta z}{2}\right) \cdot (0, 0, 1) \Delta x \Delta y + \mathbf{F}\left(x, y, z - \frac{\Delta z}{2}\right) \cdot (0, 0, -1) \Delta x \Delta y \end{aligned}$$

**Figure 8.1** The divergence of a vector field can also be defined as the total flow out through the six faces of the volume element divided by the volume of the element.



Carrying out the dot products and dividing by the volume  $\Delta V = \Delta x \Delta y \Delta z$  gives

$$\frac{\text{flow}}{\Delta V} = \frac{F_x(x + \frac{\Delta x}{2}, y, z) - F_x(x - \frac{\Delta x}{2}, y, z)}{\Delta x} + \frac{F_y(x, y + \frac{\Delta y}{2}, z) - F_y(x, y - \frac{\Delta y}{2}, z)}{\Delta y} + \frac{F_z(x, y, z + \frac{\Delta z}{2}) - F_z(x, y, z - \frac{\Delta z}{2})}{\Delta z}.$$

Taking the limit of this as the volume approaches zero gives

$$\lim_{\Delta V \rightarrow 0} \frac{\text{flow}}{\Delta V} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

This agrees with the previous definition of divergence, Eq. (4.14), which made use of the del operator. We will now use this new definition of divergence to derive Gauss' Theorem.

## 8.2 Gauss' Theorem

Gauss' theorem states that the triple integral of a divergence in a region  $R$  can be replaced by a flux integral on its surface  $S$ . The theorem states that if the following conditions hold:

- $\mathbf{F}$  is a continuous vector field with continuous derivatives throughout  $R$ ,
- $S$ , the surface of  $R$ , is piecewise-smooth with outward-pointing unit normal vector  $\hat{\mathbf{n}}$ ,

then the following statement is true:

Gauss' Theorem. $\iiint_R \nabla \cdot \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, d\sigma$	(8.3)
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Gauss' theorem is important because (1) one integral may be easier to evaluate than the other, (2) in theoretical work it may be necessary to convert one type of integral to the other, (3) it allows us to replace integrands that are expressed in derivative form with integrands that are not, and (4) it replaces an integral over an entire region of space by an integral over its surface only. In short it provides a method of integration.

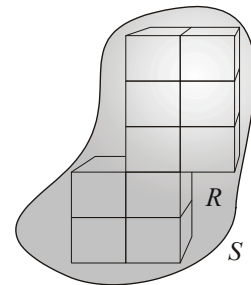
### Derivation of Gauss' Theorem

Imagine dividing the region  $R$  with surface  $S$  into many tiny rectangular solid elements (boxes) as shown in Fig. 8.2. For any given box the new definition of divergence, Eq.(8.2), states that

$$\nabla \cdot \mathbf{F} = \frac{1}{\Delta V} \cdot (\text{flow of } \mathbf{F} \text{ out through the faces}),$$

where  $\Delta V = \Delta x \Delta y \Delta z$  is the volume of the box. Now multiply both sides of this equation by  $\Delta V$  and sum up over all the boxes that make up the region  $R$ .

$$\sum_{\text{all boxes}} \nabla \cdot \mathbf{F} \Delta V = \sum_{\text{all boxes}} (\text{flow of } \mathbf{F} \text{ out through the faces}).$$



**Figure 8.2** Region  $R$  with surface  $S$  is broken up into tiny rectangular solid elements.

On the right side of the equation the sum over all the boxes can be replaced by a sum over all the faces of all the boxes. Now notice that for any two adjacent boxes the outward flow from one box is the inward flow into the other box through their common face. In other words the outward flow from one box *cancels* the outward flow from the other box through their common face. Only the flow through the exterior faces does not cancel. The result is that the right side of the equation reduces to a sum over just the exterior faces. In the limit as  $\Delta V \rightarrow 0$  the sum on the left side of the equation can be replaced by a triple integral over the volume  $R$  and the sum on the right side becomes a flux integral over the surface  $S$ . Thus the equation becomes

$$\iiint_R \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} d\sigma,$$

which is just Gauss' theorem as stated in Eq.(8.3).

### A Model of Gauss' Theorem

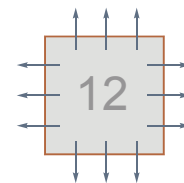
The following model will help us understand what Gauss' theorem means. Imagine a two dimensional region  $R$  with surface  $S$  broken into small squares, each with area  $\Delta A = \Delta x \Delta y = 1$ . Replace the new definition of divergence, (8.2), with this *model definition*:

$$\frac{\Delta F_x}{\Delta x} + \frac{\Delta F_y}{\Delta y} = \nabla \cdot \mathbf{F} \text{ for a small square} = \text{total flow of } \mathbf{F} \text{ out the four sides.} \quad (8.4)$$

Allow  $F_x, F_y, \nabla \cdot \mathbf{F}$ , and the flow of  $\mathbf{F}$  to take on integer values only. For each square let:

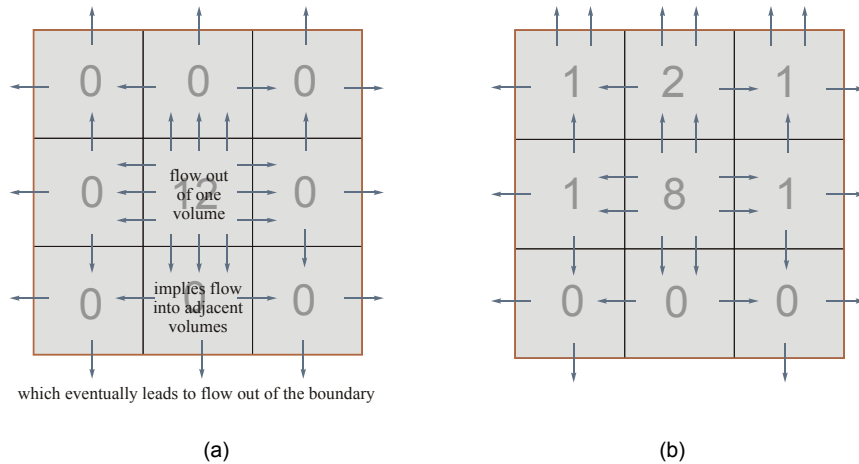
- $n$  arrows pointing to the right at the right or left edges mean that  $F_x = n$  there. Arrows to the left mean that  $F_x$  is negative there.
- $n$  arrows pointing up at the top or bottom edges mean that  $F_y = n$  there. Arrows pointing down mean that  $F_y$  is negative there.
- $n$  more arrows pointing to the right at the right edge than the left edge indicate that  $\Delta F_x / \Delta x = n$ .
- $n$  more arrows pointing up at the top edge than the bottom edge indicate that  $\Delta F_y / \Delta y = n$ .
- $\nabla \cdot \mathbf{F}$  is calculated by adding up  $\Delta F_x / \Delta x$  and  $\Delta F_y / \Delta y$  in the two previous bullets. Write this value inside the square.
- The flow of  $\mathbf{F}$  out of the square is calculated by counting how many arrows point out of the square. (Arrows pointing *into* the square count as negative outward flow.) According to Eq. (8.4) this number equals the number found in the previous bullet. Fig. 8.3 shows an example with  $n = 12$ .

**Figure 8.3** This square has  $F_x = -3$  at the left side and  $+3$  at the right side,  $F_y = -3$  at the bottom and  $+3$  at the top. It has  $\Delta F_x / \Delta x = 6$  and  $\Delta F_y / \Delta y = 6$  so the divergence is 12 (indicated by the number inside the square). The total flow outward is also 12 (indicated by the 12 arrows). Eq. (8.4) states that these numbers are always equal.



Now imagine that the square of Fig. 8.3 is the center square of a region  $R$  and is surrounded by 8 other squares whose divergence is zero as shown in Fig. 8.4(a). Applying Eq. (8.4) to each of these 8 other squares means that they must each have as many outward pointing arrows as inward pointing ones. This means that the flow out of the central square, 12, eventually reaches the boundary of  $R$  so the flow across the boundary of  $R$  is also 12.

Fig. 8.4(b) shows another example where some of the other squares making up  $R$  also have non-zero divergence. The total divergence of all the squares, 14, is the analog of the triple integral on the left-hand-side of Gauss' theorem. The total number of arrows crossing the boundary of  $R$ , also 14, is the analog of the flux integral on the right-hand-side. The fact that these numbers are equal is the content of Gauss' theorem.



**Figure 8.4** Region  $R$  with boundary  $S$  is broken into nine squares.

**(a)** A flow of 12 out of the center square eventually results in a flow of 12 across the boundary  $S$ .

**(b)** The total divergence in region  $R$  is 14. The total flow across the boundary is also 14. This is Gauss' theorem.

### 8.3 Examples using Gauss' Theorem

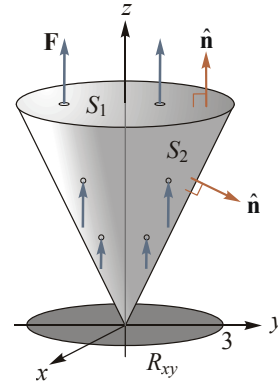
**Example 8.1:** Verify Gauss' theorem for field  $\mathbf{F} = (0, 0, z^2 + 2)$  and region  $R$  being the interior of the cone  $z \geq \sqrt{x^2 + y^2}, z \leq 3$ .

**Solution:** We must verify that  $\iiint_R \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} d\sigma$ .

- Compute the LHS. The divergence is  $\nabla \cdot \mathbf{F} = 2z$ . Use cylindrical coordinates to carry out the triple integral.

$$\begin{aligned} \text{LHS} &= \iiint_R \nabla \cdot \mathbf{F} dV = \iiint_R 2z dV \\ &= \int_0^3 \int_0^{2\pi} \int_0^z 2z r d\theta dr dz = \frac{81}{2} \pi. \end{aligned}$$

- Compute the RHS. The surface consists of 2 parts: the planar disk surface  $S_1$  defined by  $z=3$ , and the lateral conical surface  $S_2$  defined by  $z = \sqrt{x^2 + y^2}$ . Notice that the field  $\mathbf{F}$  is pointing in the  $z$  direction and gets stronger with  $z$ . Thus we expect the flux through  $S_1$  to be positive (out of the cone) and the flux through  $S_2$  to be negative (inward) and smaller. (See Fig. 8.5.)



**Figure 8.5** Verifying Gauss' theorem for a conical region.

We do the disk surface  $S_1$  first. On this surface we can evaluate  $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} d\sigma$  directly. Since  $z = 3$  on this surface the field is  $\mathbf{F} = (0, 0, 11)$ , constant, and the unit normal is  $\hat{\mathbf{n}} = \hat{\mathbf{k}} = (0, 0, 1)$  so the integrand is  $\mathbf{F} \cdot \hat{\mathbf{n}} = 11$ . The area of the disk is  $A = \pi r^2 = 9\pi$  and so we get

$$\text{Flux out through } S_1 = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dA = 11 \times 9\pi = 99\pi.$$

We do the conical surface next. We will project the surface onto the circle  $R_{xy}$  in the  $xy$  plane and evaluate  $\iint_{R_{xy}} \frac{\mathbf{F} \cdot \nabla g}{|\nabla g \cdot \hat{\mathbf{p}}|} dA$ . We can avoid square roots in the gradient if we express the surface as  $g(x, y, z) = x^2 + y^2 - z^2 = 0$ . This gives the gradient  $\nabla g = (2x, 2y, -2z)$ . Because we are projecting onto the  $xy$ -plane,  $\hat{\mathbf{p}}$  is  $\hat{\mathbf{k}}$ , and  $|\nabla g \cdot \hat{\mathbf{p}}| = |-2z| = 2z$ . Thus the flux integral is

$$\text{Flux through } S_2 = \iint_{R_{xy}} \frac{\mathbf{F} \cdot \nabla g}{|\nabla g \cdot \hat{\mathbf{p}}|} dA = \iint_{R_{xy}} \frac{(0, 0, z^2 + 2) \cdot (2x, 2y, -2z)}{|2z|} dA = \iint_{R_{xy}} (-z^2 - 2) dA.$$

Because  $R_{xy}$  is a circle this integral is easiest done in polar coordinates.

$$\iint_{R_{xy}} (-z^2 - 2) dA = \int_0^{2\pi} \int_0^3 (-r^2 - 2) r dr d\theta = -\frac{117}{2} \pi.$$

Notice that this flux is negative and smaller than the flux through  $S_1$ , as expected. Together the two flux integrals give  $\frac{81}{2} \pi$ , so the LHS equals the RHS, thus verifying Gauss' theorem.



**Example 8.2:** Verify Gauss' theorem for the field  $\mathbf{F} = (3x, 0, 0)$  and region  $R$  being a sphere of radius 3 centered on the origin.

**Solution:** Again we verify that  $\iiint_R \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} d\sigma$ .

- Compute the LHS. The divergence is  $\nabla \cdot \mathbf{F} = 3$ , i.e. constant, throughout the sphere. Since the volume of a sphere is  $\frac{4}{3}\pi r^3$  we have that

$$\text{LHS} = \iiint_R \nabla \cdot \mathbf{F} dV = 3 \times \frac{4}{3}\pi 3^3 = 108\pi.$$

- Compute the right side. The flux integral can be done directly in spherical coordinates (without projecting onto a coordinate plane). This box shows how:

**Surface integral on a sphere of radius  $a$  using spherical coordinates.** See Fig. 3.18. If the polar angle  $\varphi$  is increased by amount  $d\varphi$  and the azimuthal angle  $\theta$  is increased by amount  $d\theta$  then a rectangular element of area  $a d\varphi \times a \sin \varphi d\theta$  is swept out on the surface. Therefore the surface integral on a sphere of radius  $a$  is

$$\iint_S G(\mathbf{r}) d\sigma = \iint_S G(\theta, \varphi) a^2 \sin \varphi d\varphi d\theta.$$

The integrand  $G(\mathbf{r})$  must be converted using  $\mathbf{r} = (x, y, z) = (a \sin \varphi \cos \theta, a \sin \varphi \sin \theta, a \cos \varphi)$ .

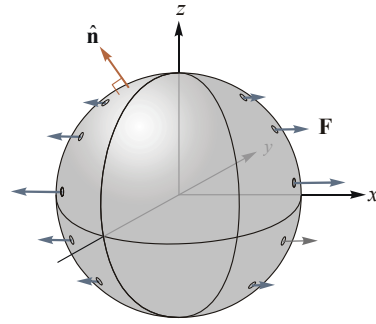
In this example the integrand is  $\mathbf{F} \cdot \hat{\mathbf{n}}$ . The field on the surface is  $\mathbf{F} = (3x, 0, 0) = (9 \sin \varphi \cos \theta, 0, 0)$ . The unit normal vector at any point on the surface is  $\hat{\mathbf{n}} = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$ . This is because any point on the surface is  $\mathbf{r} = (x, y, z) = (3 \sin \varphi \cos \theta, 3 \sin \varphi \sin \theta, 3 \cos \varphi)$  and the unit normal is exactly the same vector but normalized to have length 1. Thus  $\mathbf{F} \cdot \hat{\mathbf{n}} = 9 \sin^2 \varphi \cos^2 \theta$  and the flux integral becomes

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} d\sigma = \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} 3^2 \sin \varphi d\varphi d\theta = \int_0^{2\pi} \int_0^\pi 81 \sin^3 \varphi \cos^2 \theta d\varphi d\theta = 108\pi.$$

Thus the LHS equals the RHS, verifying Gauss' theorem.

**Note on Example 8.2:** If we *do* try calculating the flux integral by projecting onto, say, the  $xy$  plane then there are two complications. First, we have to treat the upper and lower hemispheres as separate surfaces and add the results. (In this example, though, Fig. 8.6 shows that the two hemispheres yield equal flux integrals so we can just do the upper one and double the result.) The second complication is that because the hemispheres have vertical walls at  $r = 3$  (i.e. at the equator) the projection's "correction factor" is undefined at  $r = 3$ . Thus the integral can't be done numerically on a calculator. In case you want to use this approach here are the details:

- The surface is  $g(x, y, z) = x^2 + y^2 + z^2 - 9 = 0$  so  $\nabla g = (2x, 2y, 2z)$ .
- The projection is onto the  $xy$  plane so  $\hat{\mathbf{p}} = (0, 0, 1)$ .



**Figure 8.6** Verifying Gauss' theorem for a sphere.

- The projection region  $R_{xy}$  is a circle of radius 3 in the  $xy$  plane so we will convert the double integral to polar coordinates. In the integrand we let  $x = r \cos \theta$ ,  $y = r \sin \theta$  and  $z = \sqrt{9 - x^2 - y^2}$ . The flux integral for the upper hemisphere is

$$\iint_{R_{xy}} \frac{\mathbf{F} \cdot \nabla g}{|\nabla g \cdot \hat{\mathbf{p}}|} dA = \iint_{R_{xy}} \frac{6x^2}{|2z|} dx dy = \int_0^{2\pi} \int_0^3 \frac{6(r \cos \theta)^2}{2\sqrt{9-r^2}} r dr d\theta = 3 \underbrace{\int_0^{2\pi} \cos^2 \theta d\theta}_{=\pi} \int_0^3 \frac{r^3}{\sqrt{9-r^2}} dr .$$

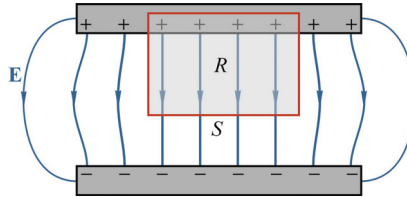
- The integral over  $r$  is improper and can't be done on the calculator. It can, however, be done using the change of variable  $u = 9 - r^2$ .

$$\int_0^3 \frac{r^3}{\sqrt{9-r^2}} dr = \frac{1}{2} \int_0^9 \frac{9-u}{\sqrt{u}} du = \left[ 9u^{1/2} - \frac{1}{3}u^{3/2} \right]_{u=0}^{u=9} = 18$$

Thus the flux through the upper hemisphere is  $54\pi$ . Doubling this gives the flux out of the whole sphere. This is the same answer that we got using spherical coordinates.



**Figure 8.7** Using Gauss' law to find the electric field of a parallel plate capacitor.



**Example 8.3:** Gauss' law states that the electric field obeys the equation  $\nabla \cdot \mathbf{E} = \rho / \epsilon_0$  where  $\rho$  is the charge density and  $\epsilon_0$  is a constant. Use Gauss' law and Gauss' theorem to find the electric field inside the charged parallel plate capacitor shown in Fig. 8.7. Suppose that the charge per unit area on the upper plate is  $\lambda$  and assume that the fringing effect at the edges can be neglected.

**Solution:** Let us write Gauss' theorem in the form  $\iiint_R \nabla \cdot \mathbf{E} dV = \iint_S \mathbf{E} \cdot \hat{\mathbf{n}} d\sigma$  and on the left side substitute in Gauss' law. This gives the formula

$$\frac{1}{\epsilon_0} \iiint_R \rho dV = \iint_S \mathbf{E} \cdot \hat{\mathbf{n}} d\sigma . \tag{8.5}$$

This is called the **integral form of Gauss' law** and it shows how the electric field on the surface of any region is related to the charge contained in that region. Let us apply (8.5) to the box-shaped region  $R$  with surface  $S$  shown in Fig. 8.7. Suppose that the bottom and top of the box have area  $A$  and that the height of the box is  $h$ . Because all the charge on the upper plate resides along its bottom edge and because the top of the box is inside the plate, the box contains an amount of charge  $\lambda A$ . Furthermore because there is no  $\mathbf{E}$  field at the top of the box and because  $\mathbf{E} \cdot \hat{\mathbf{n}}$  is zero at the four sides of the box, Eq. (8.5) reduces to

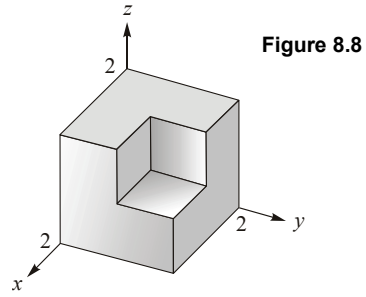
$$\frac{1}{\epsilon_0} \lambda A = \iint_{\text{box bottom}} \mathbf{E} \cdot \hat{\mathbf{n}} d\sigma = |\mathbf{E}| A , \tag{8.6}$$

where  $|\mathbf{E}|$  is the magnitude of the electric field at the bottom of the box. Eq. (8.6) shows that the electric field between the plates is straight downward and has constant magnitude  $|\mathbf{E}| = \frac{1}{\epsilon_0} \lambda$ .



**Problem Set 8.1 - Gauss' Theorem**

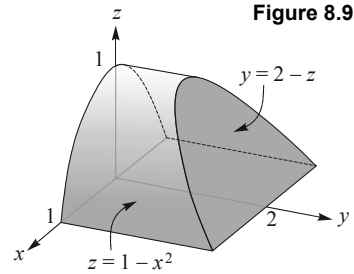
1. Find  $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} d\sigma$  if  $S$  is the surface of the  $2 \times 2 \times 2$  cube with a  $1 \times 1 \times 1$  corner missing as shown in Fig. 8.8 and  $\mathbf{F}(x, y, z) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ . (Hint: Use Gauss' theorem to replace a difficult flux integral calculation with an easy triple integral calculation.)



2. Find the outward flux of the vector field  $\mathbf{F}(x, y, z) = (x, e^{xz}, \ln(xy))$  over the unit sphere  $x^2 + y^2 + z^2 = 1$ .

3. Find the outward flux of the vector field  $\mathbf{F}(x, y, z) = z\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$  over the closed surface of the solid hemisphere  $x^2 + y^2 + z^2 = 1, z \geq 0$ .

4. Evaluate  $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} d\sigma$  if  $S$  is the outwardly oriented surface of the region bounded by the parabolic cylinder  $z = 1 - x^2$  and the planes  $z = 0, y = 0,$  and  $y + z = 2$  and if  $\mathbf{F}(x, y, z) = xy\hat{\mathbf{i}} + (y^2 + e^{xz})\hat{\mathbf{j}} + \sin(xy)\hat{\mathbf{k}}$ . See Fig. 8.9.



5. Verify Gauss' Theorem for the vector field  $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$  out of the region bounded by the paraboloid  $z = x^2 + y^2$  and the plane  $z = 4$ .

6. Verify Gauss' Theorem for the field  $\mathbf{F} = (0, 0, z^2 + 2)$  out of the solid region bounded by the cone  $z = \sqrt{x^2 + y^2}$  and the plane  $z = 3$ .

7. Show that the electric field around a point charge  $Q$  located at the origin is given by  $\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$

in spherical coordinates by applying the integral form of Gauss' law, (8.5), to a sphere of radius  $r$  centered about the origin.

8. Suppose that an infinitely long charged wire lies on the  $z$  axis and that the charge/unit length =

$\lambda$ . Show that the electric field around it is given by  $\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{\mathbf{r}}$  in cylindrical coordinates by

applying the integral form of Gauss' law, (8.5), to an infinitely long cylinder of radius  $r$  centered about the  $z$  axis.



## 8.4 Circulation and an Alternative Definition of Curl

Given a vector field  $\mathbf{F}$  and a closed curve  $C$ , the **circulation** of  $\mathbf{F}$  around the curve  $C$  is defined as the line integral

$$\text{Circulation} \equiv \oint_C \mathbf{F} \cdot d\mathbf{r}. \quad (8.7)$$

We will now give an alternate definition of the curl in terms of circulation. Suppose that  $P = (x, y, z)$  is a point in space. Then construct three different planes containing point  $P$  such that each plane has its unit normal vector  $\hat{\mathbf{n}}$  equal to one of the unit vectors,  $\hat{\mathbf{i}}$ ,  $\hat{\mathbf{j}}$ , or  $\hat{\mathbf{k}}$  (see Figs. 8.10 (a), (b), and (c) respectively.) Let  $C$  be a small closed curve that lies in a plane containing  $P$  and with unit normal vector  $\hat{\mathbf{n}}$ . Let the direction of  $C$  be related to the direction of  $\hat{\mathbf{n}}$  by the right-hand-rule and let  $\Delta\sigma$  be the area inside  $C$ . Then we claim that we can define the  $\hat{\mathbf{n}}$  component of the curl of a vector field  $\mathbf{F}$  at  $P$  in terms of the circulation like this:

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \equiv \lim_{\Delta\sigma \rightarrow 0} \frac{1}{\Delta\sigma} (\text{circulation around } C) = \lim_{\Delta\sigma \rightarrow 0} \frac{1}{\Delta\sigma} \oint_C \mathbf{F} \cdot d\mathbf{r}. \quad (8.8)$$

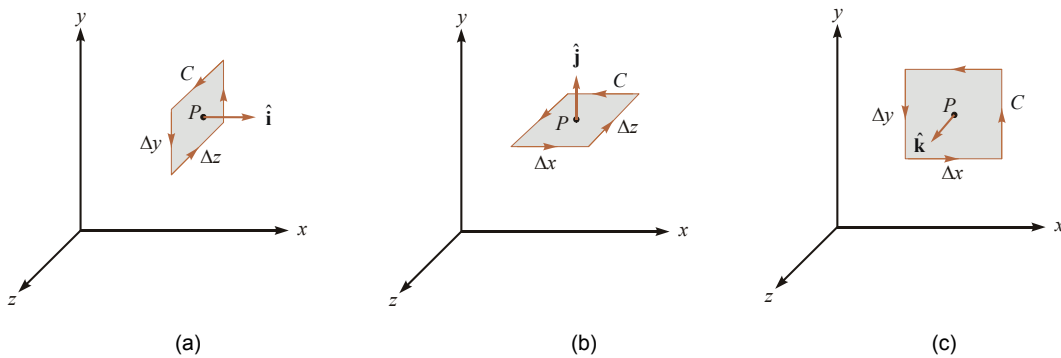
In other words to find the component of the curl in direction  $\hat{\mathbf{n}}$  we construct a vanishingly small curve  $C$  in the plane normal to  $\hat{\mathbf{n}}$ , calculate the circulation and divide by the area. For example if  $\hat{\mathbf{n}} = \hat{\mathbf{k}}$  (see Fig. 8.10(c)) then Eq. (8.8) says that the  $z$  component of the curl is

$$(\nabla \times \mathbf{F})_z = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{1}{\Delta x \Delta y} \oint_C \mathbf{F} \cdot d\mathbf{r}. \quad (8.9)$$

### Proof that the new definition of curl agrees with the old definition

Refer to Fig. 8.10(c). The circulation of  $\mathbf{F}$  around the rectangle consists of the sum of four terms of the form  $\mathbf{F} \cdot d\mathbf{r}$ ; one coming from each of the edges: right, left, top and bottom, respectively.

$$\begin{aligned} \text{circulation} = & \mathbf{F}\left(x + \frac{\Delta x}{2}, y, z\right) \cdot (0, \Delta y, 0) + \mathbf{F}\left(x - \frac{\Delta x}{2}, y, z\right) \cdot (0, -\Delta y, 0) + \\ & \mathbf{F}\left(x, y + \frac{\Delta y}{2}, z\right) \cdot (-\Delta x, 0, 0) + \mathbf{F}\left(x, y - \frac{\Delta y}{2}, z\right) \cdot (\Delta x, 0, 0) \end{aligned}$$



**Figure 8.10** (a) We can define the  $x$  component of the curl at point  $P$  as the circulation around an infinitesimally small surface element oriented as shown divided by the area of the element. (b) and (c) Similarly, we can define the  $y$  and  $z$  components of the curl with surface elements oriented as shown.

Carrying out the dot products and dividing by the area  $\Delta A = \Delta x \Delta y$  of the rectangle gives

$$\frac{\text{circulation}}{\Delta A} = \frac{F_y(x + \frac{\Delta x}{2}, y, z) - F_y(x - \frac{\Delta x}{2}, y, z)}{\Delta x} - \frac{F_x(x, y + \frac{\Delta y}{2}, z) - F_x(x, y - \frac{\Delta y}{2}, z)}{\Delta y}.$$

Taking the limit of this as the area approaches zero gives

$$\lim_{\Delta A \rightarrow 0} \frac{\text{circulation}}{\Delta A} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}.$$

This agrees with the previous definition of the  $z$  component of the curl, which was given in terms of the del operator as

$$\begin{aligned} \nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= \hat{\mathbf{i}} \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \hat{\mathbf{j}} \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \hat{\mathbf{k}} \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \end{aligned} \quad (8.10)$$

The other components of the curl can be proven in a similar fashion by looking at Figs. 8.10 (a) and (b). This completes the proof.

We will now use this new definition of the curl to derive Stokes' Theorem.

## 8.5 Stokes' Theorem

Stokes' theorem states that a certain flux integral involving a vector field  $\mathbf{F}$  on a surface  $S$  bounded by a curve  $C$  (see Fig. 8.11) can be replaced by a certain line integral on  $C$ . The theorem states that if the following conditions hold:

- $\mathbf{F}$  is a continuous vector field with continuous partial derivatives,
- $S$  is a piecewise-smooth and orientable surface with unit normal vector  $\hat{\mathbf{n}}$ ,
- $C$  is a piecewise-smooth, closed, simple curve,
- the direction of  $\hat{\mathbf{n}}$  is related to the direction of  $C$  by the right-hand-rule,

then the following statement is true:

$$\text{Stokes' Theorem.} \quad \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} \quad (8.11)$$

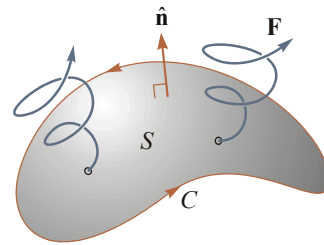


Figure 8.11 Vector field  $\mathbf{F}$ , surface  $S$  and boundary curve  $C$  for Stokes' theorem.

Stokes' theorem is important because (1) one integral may be easier to evaluate than the other, (2) in theoretical work it may be necessary to convert one type of integral to the other, (3) it allows us to replace integrands that are expressed in derivative form with integrands that are not, and (4) it replaces an integral over an entire surface by an integral over its boundary only. Thus it provides another method of integration. Stokes' theorem can be considered to be the three-dimensional generalization of Green's theorem, Eq. (6.1), which we studied in Chapter 6.

### Derivation of Stokes' Theorem

Imagine dividing the surface  $S$  into many small plane elements as shown in Fig. 8.12. For any given element the new definition of the curl, Eq. (8.8), states that

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} = \lim_{\Delta\sigma \rightarrow 0} \frac{1}{\Delta\sigma} \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where  $\Delta\sigma$  is the area of the element. Now multiply both sides of this equation by  $\Delta\sigma$  and sum up over all the elements that make up the surface  $S$ .

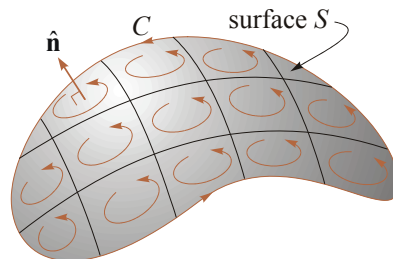
$$\sum_{\substack{\text{all surface} \\ \text{elements}}} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \Delta\sigma = \sum_{\substack{\text{all surface} \\ \text{elements}}} \oint_C \mathbf{F} \cdot d\mathbf{r} \quad (8.12)$$

On the right side of (8.12) the sum over all surface elements can be replaced by the sum over all the edges of the surface elements. Now notice that for any two adjacent elements the line integral from one element precisely cancels the line integral from the other element along their common edge. (Look at the adjacent loops in Fig. 8.12.) Only the line integral along the exterior edges does not cancel. The result is that the right side of Eq. (8.12) reduces to a sum over just the exterior edges. In the limit as  $\Delta\sigma \rightarrow 0$  the sum on the left side of (8.12) can be replaced by a surface integral over the entire surface and the sum on the right side becomes a line integral over the curve  $C$ . Thus Eq. (8.12) becomes

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r},$$

which is just Stokes' theorem as stated in Eq. (8.11).

**Figure 8.12** Stokes' theorem is derived by breaking up surface  $S$  into many small planar elements and summing up Eq. (8.8) over all of them. The sum on the LHS becomes the surface integral of Stokes' theorem and the sum on the RHS becomes the line integral around only the boundary  $C$  because the interior contributions to the line integral cancel.



### A Model of Stokes' Theorem

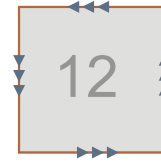
The following model can help us understand what Stokes' theorem means. (Since the model is two-dimensional it also describes Green's theorem.) Imagine a region  $R$  in the  $xy$  plane with surface  $S$  broken into small squares, each with area  $\Delta A = \Delta x \Delta y = 1$ . Replace the new definition of curl, (8.9), with this *model definition*:

$$\frac{\Delta F_y}{\Delta x} - \frac{\Delta F_x}{\Delta y} = \nabla \times \mathbf{F} \text{ for a small square} = \text{circulation around the four sides.} \quad (8.13)$$

Allow  $F_x, F_y, \nabla \times \mathbf{F}$  and the circulation of  $\mathbf{F}$  to take on integer values only. For each square let:

- $n$  arrows pointing upward at the left or right edges mean that  $F_y = n$  there. Arrows pointing downward mean that  $F_y$  is negative there.
- $n$  arrows pointing to the right at the top or bottom edges mean that  $F_x = n$  there. Arrows to the left mean that  $F_x$  is negative there.
- $n$  more arrows pointing upward at the right edge than the left edge indicate  $\Delta F_y / \Delta x = n$ .
- $n$  more arrows pointing to the right at the top edge than the bottom edge indicate that  $\Delta F_x / \Delta y = n$ .
- $\nabla \times \mathbf{F}$  is calculated by combining  $\Delta F_y / \Delta x - \Delta F_x / \Delta y$  from the two previous bullets. Write this value inside the square.
- The circulation of  $\mathbf{F}$  around the square is calculated by counting how many arrows point counterclockwise around the square. (Arrows pointing *clockwise* count as negative circulation.) According to Eq. (8.13) this number equals the number found in the previous bullet. Fig. 8.13 shows an example with  $n = 12$ .

**Figure 8.13** This square has  $F_y = -3$  at the left side and 3 at the right side,  $F_x = 3$  at the bottom and  $-3$  at the top. It has  $\Delta F_y / \Delta x = 6$  and  $\Delta F_x / \Delta y = -6$  so the curl is 12 (indicated by the number inside the square). The total circulation is also 12 (indicated by the 12 arrows). Eq. (8.13) states that these numbers are always equal.



Now imagine that the square of Fig. 8.13 is the center square of a region  $R$  and is surrounded by 8 other squares whose circulation is zero as shown in Fig. 8.14. Applying Eq. (8.13) to each of these 8 other squares means that they must each have as many clockwise pointing arrows as counterclockwise ones. This means that the circulation around the central square, 12, eventually reaches the boundary of  $R$  so the circulation around the boundary of  $R$  is also 12.

**Figure 8.14** Region  $R$  with boundary  $S$  is broken into nine squares. A circulation of 12 in the center square eventually results in a circulation of 12 around the boundary  $S$ .

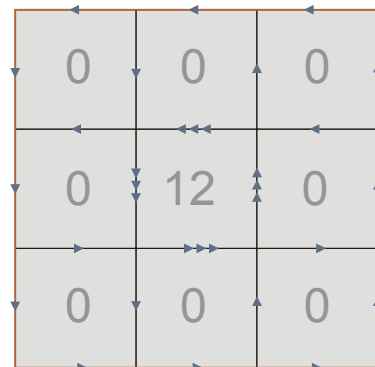
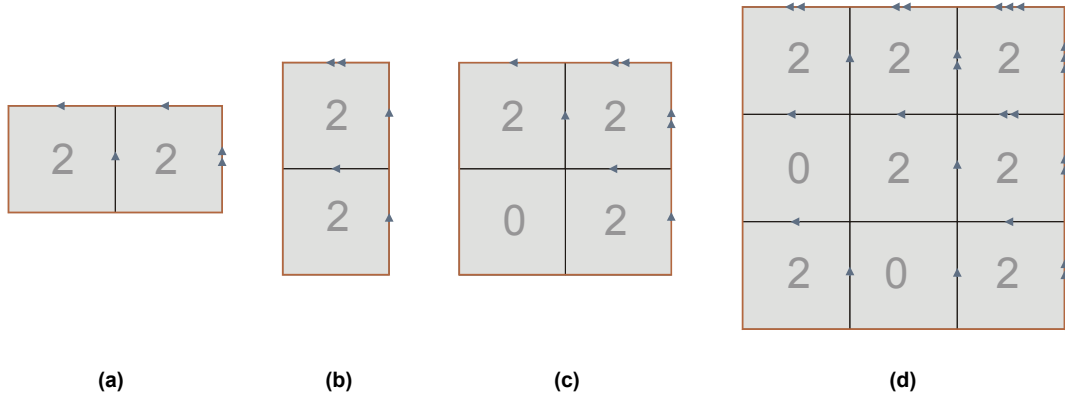


Fig. 8.15 shows four more examples where some of the other squares making up  $R$  also have non-zero circulation. In each case you should count the total curl of all the squares (the number written in the squares, the analog of the surface integral on the left-hand-side of Stokes' theorem) and count the total number of arrows pointing counterclockwise around the boundary of  $R$  (the analog of the line integral on the right-hand-side). Stokes' theorem says that these numbers are equal.



**Figure 8.15** Verifying the “model” Stokes' theorem in four examples. The curl (circulation) of a square is gotten by counting arrows in the counterclockwise direction around that square. This number is written in the square. The left side of Stokes' theorem is the sum of these numbers over all squares making up the region  $R$ . The circulation around the boundary of the region  $R$  is gotten by counting counterclockwise arrows around the boundary of  $R$ . This number is the right side of Stokes' theorem. **(a)** Both sides of Stokes' theorem give the number 4, **(b)** 4, **(c)** 6, **(d)** 14.

## 8.6 Examples using Stokes' Theorem

We now verify Stokes Theorem,  $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$ , for three examples by calculating

both sides and checking that they are equal. In simple cases like Example 8.4 the surface integral can be carried out directly. In more complicated cases like Examples 8.5 and 8.6 the surface integral is done, as in Chapter 7, by projecting onto a coordinate plane. That is, we make this replacement:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, d\sigma \rightarrow \iint_R \frac{(\nabla \times \mathbf{F}) \cdot \nabla g}{|\nabla g \cdot \hat{\mathbf{p}}|} \, dA \quad (8.14)$$

In the replacement integral,  $R$  is the region in the coordinate plane that the surface  $S$  projects onto, either  $R_{xy}$ ,  $R_{xz}$ , or  $R_{yz}$  and  $g(x, y, z) = 0$  is the equation of the surface.

**Example 8.4:** Verify Stokes' theorem in the case that the field is  $\mathbf{F} = (z^2, 5x, 0)$ , and the surface  $S$  is the square  $0 \leq x \leq 1, 0 \leq y \leq 1, z = 1$ , shown in Fig. 8.16.

**Solution:**

- On the LHS of Stokes' theorem: Calculate the curl of  $\mathbf{F}$ .

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ z^2 & 5x & 0 \end{vmatrix} = (0, 2z, 5).$$

- Let  $\hat{\mathbf{n}} = (0, 0, 1)$  and find the dot product  $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} = 5$ .
- Because this integrand is constant we can just multiply by the area of the surface, 1, and the LHS equals 5.

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, d\sigma = 5 \iint_S d\sigma = 5 \times 1 = 5$$

- On the RHS of Stokes' theorem: Calculate the contributions of the four sides:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} (1, 5x, 0) \cdot (dx, 0, 0) = \int_0^1 dx = 1$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} (1, 5, 0) \cdot (0, dy, 0) = \int_0^1 5 dy = 5$$

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} (1, 5x, 0) \cdot (dx, 0, 0) = \int_1^0 dx = -1$$

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_{C_4} (1, 0, 0) \cdot (0, dy, 0) = 0$$

... to the total line integral:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = 1 + 5 + (-1) + 0 = 5$$

The right-hand-side equals the left-hand-side so Stokes' theorem is verified.

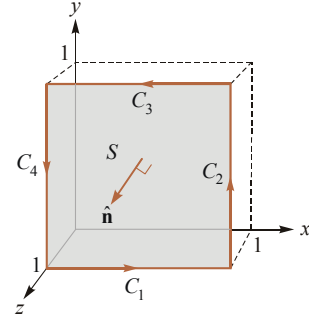


Figure 8.16

**Example 8.5:** Verify Stokes' theorem in the case that the field is  $\mathbf{F} = (e^z, e^z \sin y, e^z \cos y)$ , and the surface  $S$  is defined by  $z = y^2$ ,  $0 \leq x \leq 4, 0 \leq y \leq 2$ , as shown in Fig. 8.17.

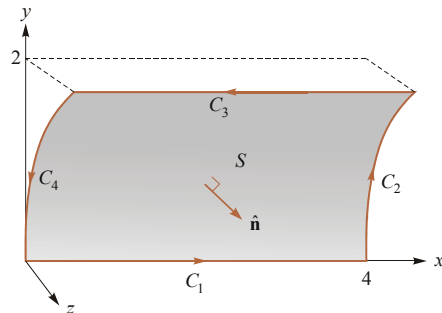


Figure 8.17

**Solution:**

- On the LHS of Stokes' theorem: define the surface as  $g(x, y, z) = z - y^2 = 0$  so  $\nabla g = (0, -2y, 1)$ .

- Calculate the curl of  $\mathbf{F}$ ,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ e^z & e^z \sin y & e^z \cos y \end{vmatrix} = (-2e^z \sin y, e^z, 0),$$

and the dot product,  $(\nabla \times \mathbf{F}) \cdot \nabla g = -2ye^z$ .

- We will project onto the  $xy$  plane so  $\hat{\mathbf{p}} = (0, 0, 1)$ ,  $|\nabla g \cdot \hat{\mathbf{p}}| = 1$ , and  $R_{xy}$  is a  $4 \times 2$  rectangle.
- Thus the surface integral on the LHS of Stokes' theorem is

$$\iint_{R_{xy}} \frac{(\nabla \times \mathbf{F}) \cdot \nabla g}{|\nabla g \cdot \hat{\mathbf{p}}|} dA = \iint_{R_{xy}} \frac{-2ye^z}{1} dA = -\int_0^4 dx \int_0^2 2ye^{y^2} dy = -4(e^4 - 1)$$

- On the RHS of Stokes' theorem: Calculate the contributions of the four sides:

- $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} (e^0, 0, 1) \cdot (dx, 0, 0) = \int_0^4 dx = 4$

- $C_2 : \mathbf{r} = (4, y, y^2), \quad 0 \leq y \leq 2$

$$\begin{aligned} \int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dy} dy &= \int_0^2 (e^z, e^z \sin y, e^z \cos y) \cdot (0, 1, 2y) dy \\ &= \int_0^2 (e^{y^2} \sin y + 2ye^{y^2} \cos y) dy = Q \quad (\text{some value that will cancel}) \end{aligned}$$

- $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} (e^4, *, *) \cdot (-dx, 0, 0) = -e^4 \int_0^4 dx = -4e^4$  (\* means doesn't matter)

- $C_4 : \mathbf{r} = (0, y, y^2), \quad y$  runs backwards from 2 down to 0.

$$\begin{aligned} \int_{C_4} \mathbf{F} \cdot \frac{d\mathbf{r}}{dy} dy &= \int_2^0 (e^z, e^z \sin y, e^z \cos y) \cdot (0, 1, 2y) dy \\ &= -\int_0^2 (e^{y^2} \sin y + 2ye^{y^2} \cos y) dy = -Q \quad (\text{the same canceling } Q \text{ value as above}) \end{aligned}$$

... to the total line integral:

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} \\ &= 4 + Q - 4e^4 - Q = -4(e^4 - 1) \end{aligned}$$

The right-hand-side equals the left-hand-side so Stokes' theorem is verified.



**Example 8.6:** Verify Stokes' theorem in the case that the field is  $\mathbf{F} = (y^2, z^2, x^2)$ , and the surface  $S$  is the portion of the paraboloid defined by  $x^2 + y^2 = z$ ,  $y \geq 0$ ,  $z \leq 1$ , as shown in Fig. 8.18.

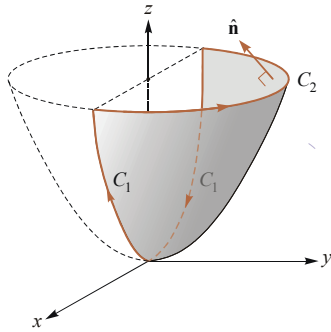


Figure 8.18

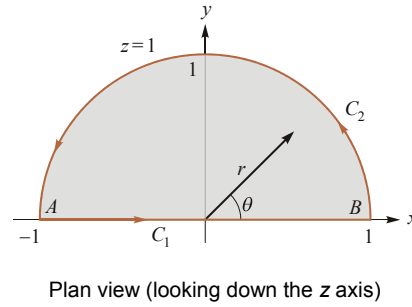


Figure 8.19

**Solution:**

- On the LHS of Stokes' theorem: define the surface as  $g(x, y, z) = z - x^2 - y^2 = 0$  and calculate  $\nabla g = (-2x, -2y, 1)$ .
- Calculate the curl of  $\mathbf{F}$ ,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ y^2 & z^2 & x^2 \end{vmatrix} = (-2z, -2x, -2y)$$

and the dot product,  $(\nabla \times \mathbf{F}) \cdot \nabla g = 4xz + 4xy - 2y$ .

- We will project onto the  $xy$  coordinate plane so  $\hat{\mathbf{p}} = (0, 0, 1)$ ,  $|\nabla \hat{\mathbf{p}} \cdot \mathbf{g}| = 1$ , and  $R_{xy}$  is the semicircle shown in Fig. 8.19.
- Thus the surface integral on the LHS of Stokes' theorem is

$$\iint_{R_{xy}} \frac{(\nabla \times \mathbf{F}) \cdot \nabla g}{|\nabla g \cdot \hat{\mathbf{p}}|} dA = \iint_{R_{xy}} \frac{4xz + 4xy - 2y}{1} dA = \iint_{R_{xy}} (4x(x^2 + y^2) + 4xy - 2y) dA$$

In the last step we used the equation of the surface to eliminate  $z$ . Because the projection region  $R_{xy}$  is a semicircle it is easiest to integrate in polar coordinates. Replacing  $x$  with  $r \cos \theta$  and  $y$  with  $r \sin \theta$ , the double integral becomes

$$\begin{aligned} &= \int_0^1 \int_0^\pi (4r^3 \cos \theta + 4r^2 \cos \theta \sin \theta - 2r \sin \theta) r dr d\theta \\ &= \int_0^\pi \left( \frac{4}{5} \cos \theta + \cos \theta \sin \theta - \frac{2}{3} \sin \theta \right) d\theta = 0 + 0 - \frac{4}{3}. \end{aligned}$$

- On the RHS of Stokes' theorem: express the total line integral as the sum of the two segments



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

- Here are the details for the individual line integral segments:

$C_1$ : See Figs. 8.18, 8.19 and 8.20. Parameterize like this:

$$C_1 : \mathbf{r} = (t, 0, t^2), \quad -1 \leq t \leq 1$$

Then

$$\begin{aligned} \int_{C_1} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt &= \int_{-1}^1 (0, t^4, t^2) \cdot (1, 0, 2t) dt \\ &= -\int_{-1}^1 2t^3 dt = 0. \end{aligned}$$

$C_2$ : See Figs. 8.18, 8.19 and 8.20. Parameterize like this:

$$C_2 : \mathbf{r} = (\cos t, \sin t, 1), \quad 0 \leq t \leq \pi$$

Then

$$\int_{C_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^\pi (\sin^2 t, 1, \cos^2 t) \cdot (-\sin t, \cos t, 0) dt = \int_0^\pi (-\sin^3 t + \cos t) dt = -\frac{4}{3}$$

The right-hand-side equals the left-hand-side so Stokes' theorem is verified.

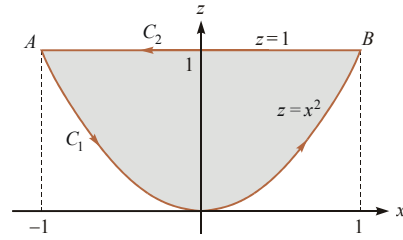


Figure 8.20 Side view (looking along the y axis).

**Example 8.7:** A solenoid is a long, thin wire wound into a tight helix. When a current  $i$  flows in the wire in the direction of the white arrows in Fig. 8.21 a magnetic field  $B$  is produced as shown. Very close to any individual winding of wire the magnetic field wraps azimuthally around the wire as described in Example 6.3. However farther away from the individual windings their magnetic fields add to produce a strong, nearly uniform magnetic field down the axis of the solenoid and a weak magnetic field outside the solenoid. The longer the solenoid the stronger and more uniform the magnetic field inside the solenoid and the weaker the magnetic field outside.

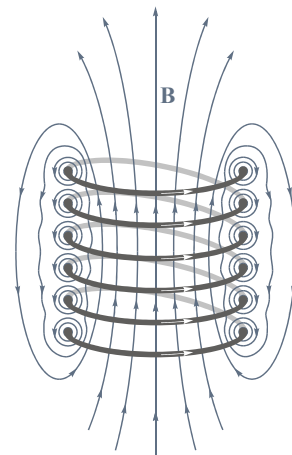


Figure 8.21 Magnetic field near a current-carrying solenoid.

Use Ampère's law,  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ , together with Stokes' theorem to calculate the magnetic field inside and around an infinitely long solenoid. See Example 6.3 and Chapter 9 for more information on Ampère's law.

**Solution:** Let the solenoid lie on the  $z$  axis as shown in Fig. 8.22 where the  $z$  axis points upward,  $y$  axis points to the right and the  $x$  axis is out of the page. On the  $x = 0$  plane the current  $i$  in the solenoid comes straight out of the page repeatedly on the left side, once for each winding (indicated by the  $\odot$  symbols) and goes back into the page repeatedly on the right side (indicated by the  $\otimes$  symbols). Because the solenoid is infinitely long the magnetic field  $\mathbf{B}$  cannot vary with  $z$  even though it points in the  $z$  direction.

Now imagine a rectangular surface  $S$  with boundary curve  $C$  laying in the  $x = 0$  plane. Apply Stokes' theorem,

$$\iint_S (\nabla \times \mathbf{B}) \cdot \hat{\mathbf{n}} \, d\sigma = \oint_C \mathbf{B} \cdot d\mathbf{r},$$

to it. Use Ampère's law,  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ , to write it as

$$\iint_S \mu_0 \mathbf{J} \cdot \hat{\mathbf{n}} \, d\sigma = \oint_C \mathbf{B} \cdot d\mathbf{r}. \quad (8.15)$$

If we place the surface  $S$  as shown in Fig. 8.22 so that  $N$  windings pass through it then the surface integral on the left side of Stokes' theorem reduces to  $\mu_0 N i$ . Furthermore the line integral on the right side of Stokes' theorem has contributions only from the left and right edges of  $C$  because along the top and bottom edges  $d\mathbf{r}$  and  $\mathbf{B}$  are normal (perpendicular). Thus we get

$$\mu_0 N i = \int_{\text{left edge of } C} \mathbf{B} \cdot d\mathbf{r} + \int_{\text{right edge of } C} \mathbf{B} \cdot d\mathbf{r}, \quad (8.16)$$

where  $N$  is the number of windings enclosed by  $C$ . Now consider what Eq. (8.16) says when the surface moved left or right to one of three different positions:

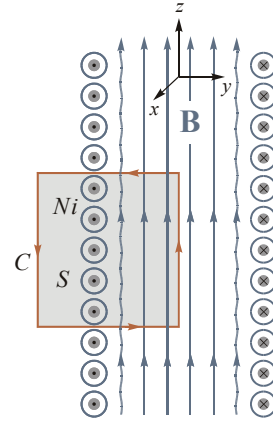
1. The surface completely inside the solenoid. In this case no windings pass through  $S$  so the left side of (8.16),  $\mu_0 N i$ , is zero implying that the two line integrals on the right side of (8.16) are equal and opposite. This means that  $\mathbf{B}$  is constant inside the solenoid.
2. The surface completely outside the solenoid. Again no windings pass through  $S$ , again implying the two line integrals are equal and opposite. But this time one edge of  $C$  can be at infinity where the magnetic field must be zero. This means that  $\mathbf{B}$  is zero outside the solenoid.
3. The surface as shown in Fig. 8.22 with  $N$  windings passing through it. In this case the left side of (8.16),  $\mu_0 N i$ , is non-zero. Furthermore from the previous case we know that the line integral from the edge outside the solenoid is zero. Thus (8.16) reduces to

$$\mu_0 N i = \int_{\text{edge inside solenoid}} \mathbf{B} \cdot d\mathbf{r} = |\mathbf{B}| L,$$

where  $L$  is the length of the inside edge of  $C$ . Thus the magnitude of  $\mathbf{B}$  inside the solenoid is

$$|\mathbf{B}| = \frac{\mu_0 N i}{L},$$

a constant that depends only on the winding density,  $N/L$ , and the current  $i$ .

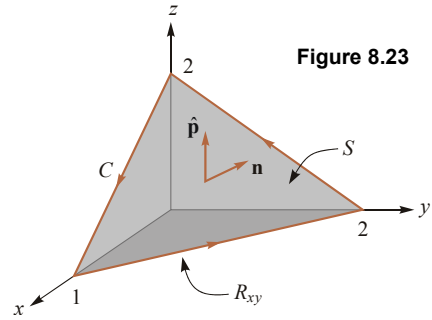


**Figure 8.22** Using Stokes' theorem to find the magnetic field  $\mathbf{B}$  inside a solenoid. The total current through the surface  $S$  is  $Ni$ . The entire contribution to the line integral is from the right edge of  $C$ .

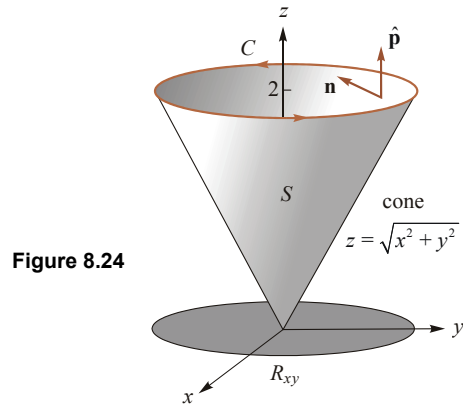


**Problem Set 8.2 - Stokes' Theorem**

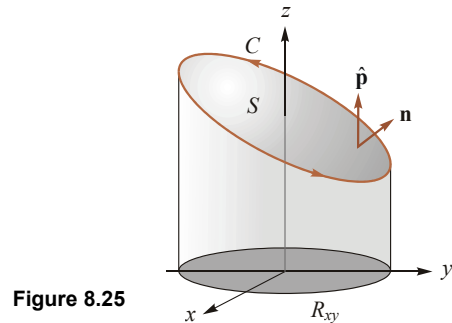
1. Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , if  $\mathbf{F}(x, y, z) = xz\hat{\mathbf{i}} + xy\hat{\mathbf{j}} + 3xz\hat{\mathbf{k}}$ , and  $C$  is the boundary of the portion of the plane  $2x + y + z = 2$  in the first octant, traversed in the direction shown in Fig. 8.23. (Hint: Use Stokes' theorem to replace the calculation of the line integral with an easier calculation of a surface integral.)



2. Find the circulation of the field  $\mathbf{F} = (x^2 - y)\hat{\mathbf{i}} + 4z\hat{\mathbf{j}} + x^2\hat{\mathbf{k}}$  around the curve  $C$  in which the plane  $z = 2$  meets the cone  $z = \sqrt{x^2 + y^2}$  with orientation shown in Fig. 8.24.



3. Use Stokes' theorem to evaluate the line integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , for  $\mathbf{F}(x, y, z) = -y^2\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ , and where  $C$  is the curve of intersection of the plane  $y + z = 2$  and the cylinder  $x^2 + y^2 = 1$ , with  $C$  oriented as shown in Fig. 8.25.



4. The solenoid in Example 8.7 can be bent and the ends joined to form a **toroidal solenoid** as shown in Fig. 8.26. Use Stokes' theorem together with Ampere's law  $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$  to show that the magnetic field is  $\mathbf{B} = \frac{\mu_0 N i}{2\pi r} \hat{\boldsymbol{\theta}}$  inside the solenoid (in cylindrical coordinates) and zero outside the solenoid. (Hint: Apply a disk-shaped surface  $S$  with various radii  $r$ .) This type of solenoid has the great advantage that the magnetic field is confined and won't affect other components in an electric circuit.

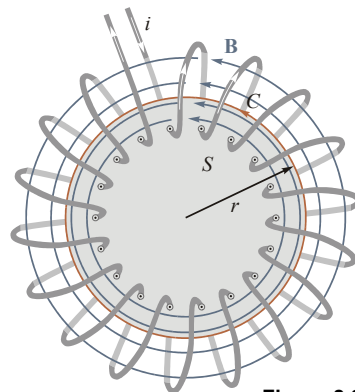


Figure 8.26

## Answers, Chapter 8 – Gauss' and Stokes' Theorems

### Problem Set 8.1 - Gauss' Theorem, page 150

1. 21                      2. volume of sphere =  $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi$                       3.  $\pi/2$                       4. 184/35
5.  $16\pi$  (Top:  $12\pi$ , Side:  $4\pi$ )                      6.  $81\pi/2$  (Top:  $99\pi$ , Side:  $-117\pi/2$ )

### Problem Set 8.2 - Stokes' Theorem, page 161

1. -1                      2.  $4\pi$                       3.  $\pi$