1. Vectors and Curves in Space

The *xyz* Rectangular Coordinate System. The familiar *xy* rectangular coordinate system used in two dimensional analytic geometry can be extended to three dimensions. We use three axes, *x*, *y* and *z*, all mutually perpendicular, as shown in Fig. 1.1.

Note that if we draw an *x* axis and a *y* axis then there remain two possible directions for the *z* axis. One choice results in a so-called **right-handed coordinate system** and the other left-handed. Suppose that we curl the fingers of our right hand from the *x* axis toward the *y* axis. If our thumb then points along the *z* axis then this is called a right-handed coordinate system. In this book we will always use a right-handed coordinate system.

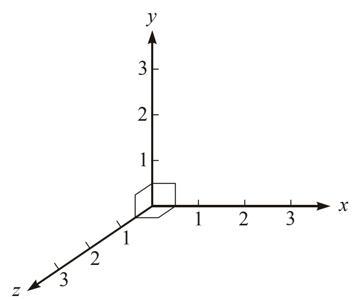


Figure 1.1 The *x*, *y* and *z* axes in a right-handed coordinate system. The *z* axis is coming out of the page.

We use an **ordered triple** (x, y, z) to denote the location of any point in three-dimensional space. The **distance** d between any two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is given by the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{1.1}$$

Example 1.1: Find the distance between points P_1 and P_2 in Fig. 1.2.

Solution: The distance is

$$\begin{aligned} d &= \sqrt{(2-(-1))^2 + (4-2)^2 + ((-1)-2)^2} \\ &= \sqrt{(3)^2 + (2)^2 + (-3)^2} \\ &= \sqrt{22} \\ &= 4.69 \end{aligned}$$

Fig. 1.2 shows how Eq. (1.1) results from applying Pythagoras' theorem twice: first to the triangle lying flat in the *xz* plane and then to the upright triangle whose base is the hypotenuse of the first triangle.

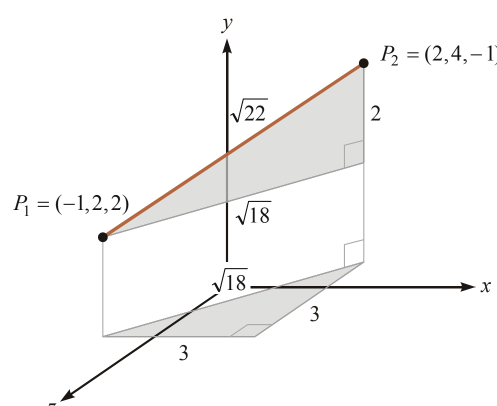


Figure 1.2 Calculating the distance between two points.

Surfaces and Curves. In a three-dimensional space a single equation relating *x*, *y* and *z* defines a surface. Two simultaneous equations relating *x*, *y* and *z* define a curve. Three simultaneous equations relating *x*, *y* and *z* define a point. Fig. 1.3 shows some examples of surfaces and curves.

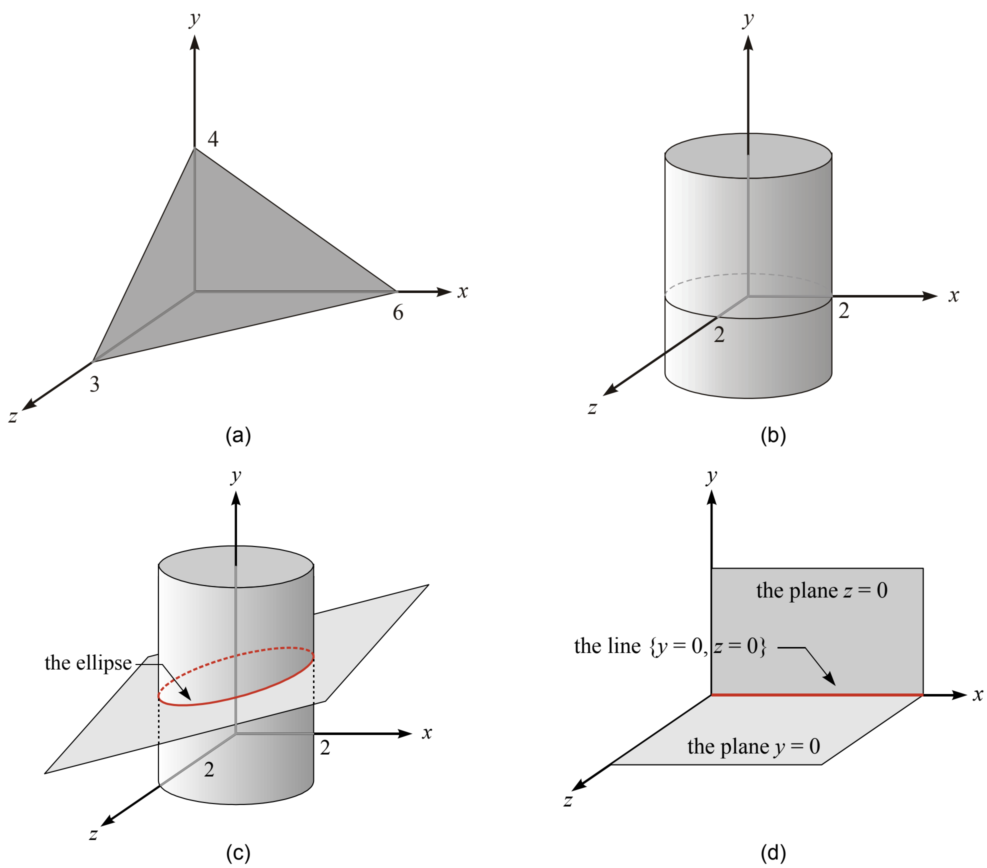


Figure 1.3
(a) The plane $2x + 3y + 4z = 12$. (This plane extends to infinity but only the region where *x*, *y* and *z* are all positive has been drawn.)
(b) The cylinder $x^2 + z^2 = 2^2$. In a two-dimensional space this would merely be a circle but since the equation does not specify *y* it is arbitrary resulting in an infinite cylinder parallel to the *y* axis.
(c) The intersection of the cylinder $x^2 + z^2 = 2^2$ with the plane $y = x + 4$ is an ellipse.
(d) The intersection of the planes $y = 0$ and $z = 0$ is the entire *x* axis.

1.1 Vectors

A **scalar** is a quantity that has only a magnitude; a **vector** is a quantity that has direction as well as magnitude. An example of a scalar is temperature. Examples of vectors are force, position, velocity and acceleration.

A vector is represented geometrically by an arrow.

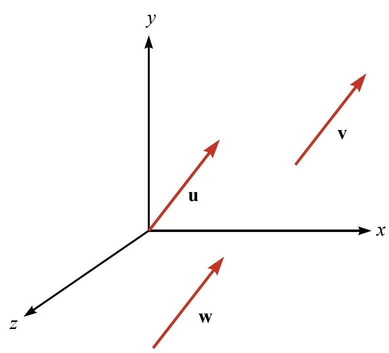
- The direction of the arrow is the direction of the vector,
- The length of the arrow *represents or is proportional to* the magnitude of the vector.

In written material vectors are denoted in various ways.

- In printed material they are often denoted by a name written in bold type, for example **v**.
- In handwritten material they are often denoted by a name with an arrow over it, e.g. $\vec{v}$.
- A vector from point *P* to point *Q* may be denoted $\overrightarrow{PQ}$.

Equality of Vectors. Two vectors are defined to be equal if they have the same magnitude and the same direction. (Their location is irrelevant.)

Figure 1.4 Vectors **u**, **v** and **w** are all equal even though they are applied at different locations. Vector **u**, with its tail at the origin, is said to be in the "standard position".

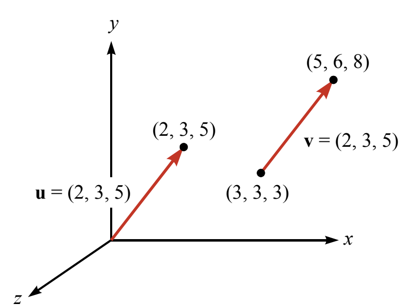


Components of a Vector. Suppose that we are given two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. The vector $\overrightarrow{PQ}$ whose tail is at *P* and whose head is at *Q* is represented algebraically by the ordered triple of numbers,

$$\overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1). \tag{1.2}$$

The number $x_2 - x_1$ is called the ***x* component** of the vector, the number $y_2 - y_1$ is the ***y* component** and $z_2 - z_1$ is the ***z* component**. These three numbers describe how far we have to go to get from the tail to the head of the vector. Thus two vectors are equal if they have the same components. (Again their location is irrelevant.)

Figure 1.5 Vectors **u** and **v** have the same components and are therefore equal even though they are applied at different locations.



The length of vector $\overrightarrow{PQ}$ is denoted $|\overrightarrow{PQ}|$ and is simply the distance between its endpoints,

$$|\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\ . \tag{1.3}$$

Multiplication of a Vector by a Scalar. The multiplication of a vector **v** by a scalar *k* is denoted *k***v** and is defined geometrically to result in a vector *k* times as long as **v** but still pointing in the same direction as **v**. Algebraically if $\mathbf{v} = (x, y, z)$ then $k\mathbf{v} = (kx, ky, kz)$. Thus each component is multiplied by *k*. Note that if *k* is a negative number then *k***v** is *k* times as long but points in the *opposite direction* of **v**.

Unit Vectors. A **unit vector** is any vector that has magnitude or length equal to 1. It is denoted by placing a circumflex or "hat" over it, thus: $\hat{\mathbf{v}}$. Three special unit vectors are $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ which are defined as unit vectors pointing along the *x*, *y* and *z* axes. In terms of components:

$$\begin{aligned} \hat{\mathbf{i}} &= (1,0,0) \\ \hat{\mathbf{j}} &= (0,1,0) \\ \hat{\mathbf{k}} &= (0,0,1) \end{aligned} \tag{1.4}$$

For any vector **v** we can construct a unit vector $\hat{\mathbf{v}}$ that points in the same direction by calculating its length, |**v**|, and then scalar multiplying vector **v** by 1/|**v**|. That is,

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}. \tag{1.5}$$

This process is called **normalizing v**. If we solve (1.5) for **v** we get

$$\mathbf{v} = |\mathbf{v}|\, \hat{\mathbf{v}}\ . \tag{1.6}$$

This has the nice interpretation that any vector can be expressed as the product of its magnitude and its corresponding unit vector.

Example 1.2: Construct the unit vector $\hat{\mathbf{v}}$ pointing in the same direction as the vector **v** pointing from *P* to *Q*, where $P = (3,2,2)$ and $Q = (5,1,-1)$.

Solution: First find the components of the vector **v**.

$$\mathbf{v} = (2,-1,-3)$$

Next find its length.

$$|\mathbf{v}| = \sqrt{2^2 + (-1)^2 + (-3)^2} = \sqrt{14} = 3.74$$

Now divide the vector by its length. The result is the unit vector

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \tfrac{1}{3.74}(2,-1,-3) = (0.53,-0.27,-0.80)$$

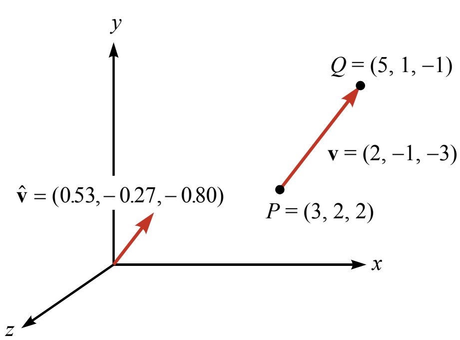


Figure 1.6 Constructing a unit vector.

Addition of Vectors. Two vectors **u** and **v** are added geometrically by placing the tail of one vector, say **v**, at the head of the other vector, **u**. The resultant or sum is a vector whose head is at the head of **v** and whose tail is at the tail of **u**.

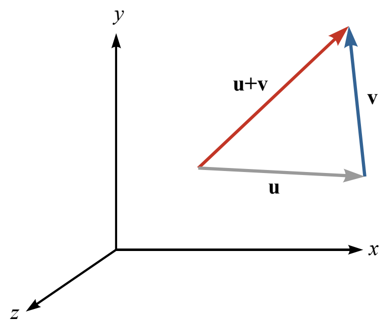


Figure 1.7 The addition of the vectors **u** and **v** produces the resultant **u + v**.

Two vectors are added algebraically by adding their corresponding components. For example if $\mathbf{u} = (5,1,1)$ and $\mathbf{v} = (2,4,-1)$ then $\mathbf{u}+\mathbf{v} = (7,5,0)$.

Subtraction of Vectors. A vector **v** is subtracted from a vector **u** geometrically by placing their tails together. Then the difference **u** − **v** is a vector whose head is at the head of **u** and whose tail is at the head of **v**. Another way to do the subtraction is to construct the vector −**v** and then add $\mathbf{u}+(-\mathbf{v})$.

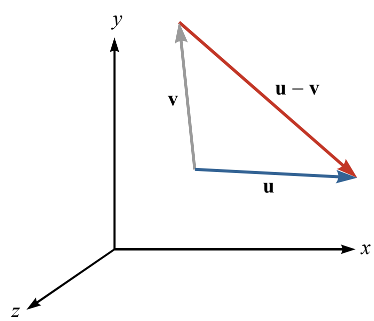


Figure 1.8 The subtraction of the vector **v** from the vector **u** produces the resultant **u** – **v**.

Two vectors are subtracted algebraically by subtracting the corresponding components. For example if $\mathbf{u} = (5,1,1)$ and $\mathbf{v} = (2,4,-1)$ then $\mathbf{u}-\mathbf{v} = (3,-3,2)$.

Resolution of a Vector into Components. Any vector $\mathbf{v} = (x, y, z)$ can be expressed as the sum of three component vectors, each pointing along one of the axes. In terms of components,

$$\mathbf{v} = \mathbf{v}_x + \mathbf{v}_y + \mathbf{v}_z$$

where $\mathbf{v}_x = (x,0,0)$, $\mathbf{v}_y = (0,y,0)$, and $\mathbf{v}_z = (0,0,z)$. In terms of unit vectors,

$$\mathbf{v} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}.$$

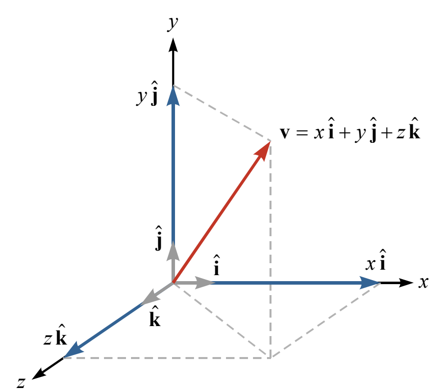


Figure 1.9 Resolution of a vector $\mathbf{v} = (x, y, z)$ into three component vectors, each parallel to one of the axes.

Parametric Lines

A straight line is uniquely determined by a point on the line and the direction of the line. Using this fact we can define a line parametrically (i.e. we can use a parameter that determines where along the line we are).

Let **r** be a point on the line and let **v** be the direction of the line. Then for any value of t,

$$\mathbf{u} = \mathbf{r} + t\,\mathbf{v} \qquad (1.7)$$

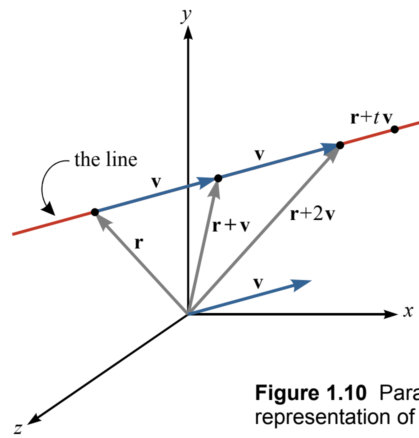


Figure 1.10 Parametric representation of a line.

is also on the line. By choosing various values of t we can move from point to point along the line. Eq. (1.7) is called the parametric equation for the line. If **u** is expressed in component form, (x, y, z), then (1.7) can be written in component form as the set of parametric equations:

$$\begin{cases} x = r_1 + t\,v_1 \\ y = r_2 + t\,v_2 \\ z = r_3 + t\,v_3 \end{cases} \qquad (1.8)$$

Example 1.3: Find where the line connecting the points $\mathbf{r} = (-1, 2, 10)$ and $\mathbf{s} = (2, 0, 5)$ intersects the plane $2x - 2y - z = 14$ shown in Fig. 1.11.

Solution: Construct the vector **v** = **s** – **r**.

$$\mathbf{v} = (3, -2, -5)$$

Now write the vector **u** = **r** + t **v** in component form as shown in Eq. (1.8) above:

$$\begin{cases} x = -1 + 3t \\ y = 2 - 2t \\ z = 10 - 5t \end{cases}$$

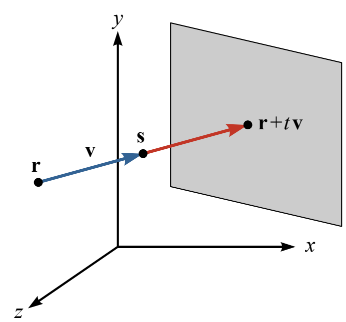


Figure 1.11 Intersection point of a line with a plane.

Now substitute these values for x, y and z into the equation for the plane. The result is

$$2(-1 + 3t) - 2(2 - 2t) - (10 - 5t) = 14.$$

Solving for t gives $t = 2$. Substituting this back in the parametric equations for x, y and z gives

$$\begin{cases} x = 5 \\ y = -2. \\ z = 0 \end{cases}$$

Thus the intersection point is $(5, -2, 0)$.

Problem Set 1.1 - Vectors

1. If $P=(4,2,8)$ and $Q=(-3,2,2)$ are two points in three-space, then:

(a) Calculate the distance between P and Q.

(b) Construct a vector that points from P to Q.

(c) Construct the unit vector that points from P to Q.

2. If $\mathbf{u}=(3,1,4)$ and $\mathbf{v}=(2,2,-3)$ are two vectors in three-space then calculate:

(a) $|\mathbf{u}|$ (b) $|3\mathbf{v}|$

(c) $\mathbf{u}+\mathbf{v}$ (d) $\mathbf{u}-\mathbf{v}$

(e) $3\mathbf{u}-2\mathbf{v}$ (f) $\hat{\mathbf{u}}$

(g) $\hat{\mathbf{v}}$

3. Find the point where the line through the point $\mathbf{r}=(4,1,-10)$ and in the direction $\mathbf{s}=(2,0,5)$ intersects the plane $2x-2y-z=14$.

4. Find the point where the line connecting the points $\mathbf{r}=(-1,2,5)$ and $\mathbf{s}=(1,1,1)$ intersects the plane $3x+y+z=1$.

5. Determine whether or not the three given points are collinear.

(a) $P = (0, -2, 4)$, $Q = (1, -3, 5)$, $R = (4, -6, 8)$

(b) $P = (6, 7, 8)$, $Q = (3, 3, 3)$, $R = (12, 14, 16)$

6. The direction of a vector is often given by its *direction cosines*. To define these, let $\mathbf{A}=a\,\hat{\mathbf{i}}+b\,\hat{\mathbf{j}}+c\,\hat{\mathbf{k}}$ be a vector, and let α, β, and γ be the three *direction angles* ($\le \pi$) that $\mathbf{A}$ makes respectively with $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. Then the direction cosines are the cosines of these direction angles.

(a) Show that $\dfrac{\mathbf{A}}{|\mathbf{A}|}=\cos\alpha\,\hat{\mathbf{i}}+\cos\beta\,\hat{\mathbf{j}}+\cos\gamma\,\hat{\mathbf{k}}$.

(b) Find the *direction cosines* of $\mathbf{A}=-\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}+2\,\hat{\mathbf{k}}$.

(c) Find the *direction angles* α, β, and γ.

7. Prove using vector methods (i.e. without components) that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. (Call the two sides **a** and **b**.)

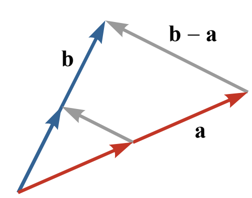


Figure 1.12

1.2 Dot Product

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors then their dot product, $\mathbf{u} \cdot \mathbf{v}$, is defined as the *number*

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 . \tag{1.9}$$

For example the dot product of vectors $\mathbf{u} = (1, 5, -2)$ and $\mathbf{v} = (1, -3, -3)$ is

$$\mathbf{u} \cdot \mathbf{v} = (1)(1) + (5)(-3) + (-2)(-3) = -8 .$$

Some of the properties of the dot product are:

- The commutative property.

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \tag{1.10}$$

- The distributive property.

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \tag{1.11}$$

- The dot product of the special unit vectors:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \quad \text{and} \quad \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 . \tag{1.12}$$

- The dot product of a vector with itself equals the length squared of the vector.

$$\mathbf{v} \cdot \mathbf{v} = (v_1, v_2, v_3) \cdot (v_1, v_2, v_3) = {v_1}^2 + {v_2}^2 + {v_3}^2 = |\mathbf{v}|^2 \tag{1.13}$$

- The geometric interpretation: See Fig. 1.13. If θ is the angle formed when vectors **a** and **b** are placed with their tails together then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta . \tag{1.14}$$

Proof
To prove that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$ we first apply the cosine law to the triangle shown in Fig. 1.14. The cosine law states that

$$|\mathbf{b} - \mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta .$$

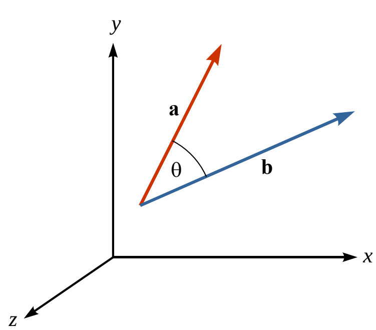


Figure 1.13 The dot product of vectors **a** and **b** equals $|\mathbf{a}||\mathbf{b}|\cos\theta$.

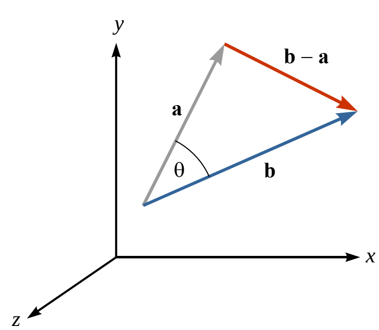


Figure 1.14 Applying the cosine law.

Now apply (1.13) to the first three terms and then use properties (1.11) and (1.10) on the left side:

$$(\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a}) = \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2|\mathbf{a}||\mathbf{b}|\cos\theta$$

$$\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2|\mathbf{a}||\mathbf{b}|\cos\theta$$

After simplifying this reads

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

which is what we set out to prove.

An important application of (1.14) is the calculation of the angle between vectors.

Example 1.4: Find the angle between the vectors $\mathbf{u} = (1,5,-2)$ and $\mathbf{v} = (1,-3,-3)$.

Solution: First calculate the dot product and lengths of **u** and **v**:

$$\mathbf{u}\cdot\mathbf{v} = (1)(1)+(5)(-3)+(-2)(-3) = -8$$

$$|\mathbf{u}| = \sqrt{1+25+4} = 5.48$$

$$|\mathbf{v}| = \sqrt{1+9+9} = 4.36$$

Then substitute them into (1.14).

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \;\rightarrow\; \theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}(-0.335) = 110^\circ$$

Orthogonality. An immediate and very important consequence of the equation $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$ is that vectors **a** and **b** are **orthogonal** (or **perpendicular** or **normal** to each other) if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.

Example 1.5: Find the equation of the plane containing the point (1, 0, 10) and which is normal (perpendicular) to vector (5, 2, 1).

Solution: Let (x, y, z) denote any point in the plane. Then the vector pointing from (1, 0, 10) to (x, y, z) must be perpendicular to the vector (5, 2, 1) as shown in Fig. 1.15. This means that their dot product is zero.

$$(5,2,1)\cdot((x,y,z)-(1,0,10)) = 0$$

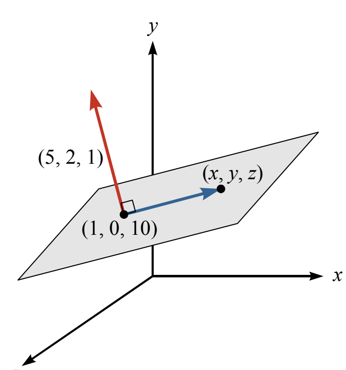


Figure 1.15 Finding the equation of a plane perpendicular to a vector.

Expanding and simplifying gives

$$(5,2,1)\cdot(x-1,y,z-10)=0$$

or

$$5(x-1)+2(y)+1(z-10)=0$$

or

$$5x+2y+z=15.$$

This is the equation of the plane. It is useful to note that the coefficients of x, y and z in this equation (namely 5, 2 and 1) correspond to the components of the normal vector, namely (5, 2, 1).

Example 1.6: The work, W, done when moving an object from point a to point b is defined as the force applied in the direction of the motion times the distance moved. Using the dot product this can be expressed as

$$W=\mathbf{F}\cdot\mathbf{s}. \qquad (1.15)$$

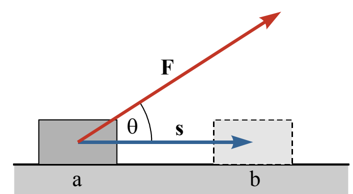
Problem: Show that if **F** is constant throughout space (say in the downward direction like the force due to gravity) then the work done to move an object around a **closed path** (one that ends where it began) is zero.

Solution: The complete path **s** can be considered to consist of a large number of straight line (vector) segments $\mathbf{s}_1,\mathbf{s}_2,\cdots,\mathbf{s}_n$. The total work done can then be written as a sum.

$$W=\sum_{i=1}^{n}\mathbf{F}_i\cdot\mathbf{s}_i$$

Since the force $\mathbf{F}_i$ on each segment is the same, it can be factored out of the sum. Also since the path is closed the sum of the vectors is zero and we get the result we set out to prove.

$$W=\mathbf{F}\cdot\sum_{i=1}^{n}\mathbf{s}_i=\mathbf{F}\cdot\mathbf{0}=0$$



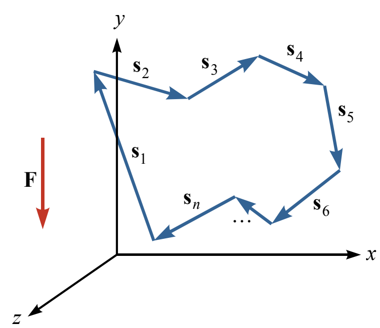


Figure 1.16 (a) The work done by the force is defined as $W=\mathbf{F}\cdot\mathbf{s}$. (b) The work done around a closed path.

Resolving a Vector into Components Parallel and Perpendicular to another Vector.

Referring to Fig. 1.17, we wish to resolve a vector **v** into a component $\mathbf{v}_{||}$, parallel to, and a component $\mathbf{v}_{\perp}$, perpendicular to, another vector **u**.

The first thing we will do is to convert vector **u** to unit vector $\hat{\mathbf{u}}$ by dividing it by its own length.

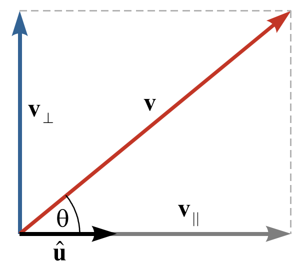


Figure 1.17 Resolving v into $\mathbf{v}_{||}$ and $\mathbf{v}_{\perp}$.

Now notice that $\mathbf{v}\cdot\hat{\mathbf{u}} = |\mathbf{v}||\hat{\mathbf{u}}|\cos\theta = |\mathbf{v}|\cos\theta$ and that this is just the length of $\mathbf{v}_{||}$. To construct the actual vector $\mathbf{v}_{||}$ we need its direction as well as its length. But its direction is $\hat{\mathbf{u}}$. Thus we see that

$$\mathbf{v}_{||} = |\mathbf{v}_{||}|\,\hat{\mathbf{u}} = (\mathbf{v}\cdot\hat{\mathbf{u}})\,\hat{\mathbf{u}}. \qquad (1.16)$$

To construct the vector $\mathbf{v}_{\perp}$ simply use the fact that $\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp}$. Solving for $\mathbf{v}_{\perp}$ gives

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{||} = \mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{u}})\,\hat{\mathbf{u}}. \qquad (1.17)$$

Example 1.7: Resolve the two-dimensional vector **v** = (4, 2) into components parallel and perpendicular to the unit vector $\hat{\mathbf{i}}$.

Solution: The parallel component is

$$\mathbf{v}_{||} = (\mathbf{v}\cdot\hat{\mathbf{i}})\,\hat{\mathbf{i}} = (4,2)\cdot(1,0)\,\hat{\mathbf{i}} = 4\,\hat{\mathbf{i}},$$

and the perpendicular component is

$$\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{||} = (4,2) - 4\,\hat{\mathbf{i}} = (0,2) = 2\,\hat{\mathbf{j}},$$

as we probably expected.

Example 1.8: Calculate the distance *d* between the point $(5,2,-1)$ and the plane $2x+1y+4z=10$.

Solution: By the distance we mean the length of the line segment **PR** which is normal (perpendicular) to the plane (see Fig. 1.18). The distance *d* can be found by constructing the unit vector $\hat{\mathbf{n}}$ normal to the plane and then calculating $\overrightarrow{QP}\cdot\hat{\mathbf{n}}$ which equals $|\overrightarrow{QP}|\cos\theta$ which equals *d*, except for possibly a minus sign. The point **Q** can be any point on the plane.

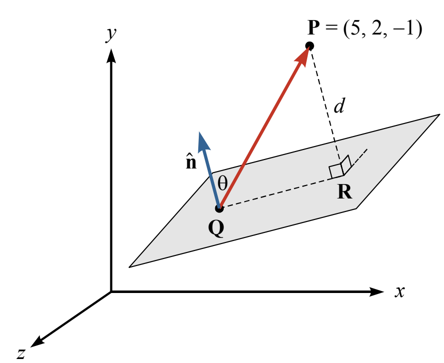


Figure 1.18 Finding the distance from a point to a plane.

First, we notice that (2, 1, 4) is one vector normal to the plane. Dividing it by its own length yields $\hat{\mathbf{n}} = \frac{1}{\sqrt{21}}(2,1,4)$.

Next we choose **Q** to be any point in the plane, say **Q** = (5, 0, 0), and form the vector $\overrightarrow{QP} = (0,2,-1)$.

Finally we calculate the dot product $\overrightarrow{QP}\cdot\hat{\mathbf{n}} = \frac{1}{\sqrt{21}}(0,2,-1)\cdot(2,1,4) = -\frac{2}{\sqrt{21}} = -0.436$. This means that the distance is $d = 0.436$.

Problem Set 1.2 - Dot Product

1. Calculate $\mathbf{u}\cdot\mathbf{v}$ if $\mathbf{u} = (3,1,4)$ and $\mathbf{v} = (2,2,-3)$.

2. Calculate $\mathbf{r}\cdot\mathbf{s}$ if $\mathbf{r} = (4,1,-10)$ and $\mathbf{s} = (2,0,5)$.

3. Find the angle between the vectors $\mathbf{r} = (4,1,-10)$ and $\mathbf{s} = (2,0,5)$.

4. Find the angle between the vectors $\mathbf{r} = (4,-3,5)$ and $\mathbf{s} = (-1,0,-1)$.

5. Find the equation of the plane containing the point (3, 3, –5) and perpendicular to the vector (1, 2, 1).

6. Resolve the vector $\mathbf{v} = (2,2,-3)$ into components parallel and perpendicular to the vector $\mathbf{u} = (3,1,4)$.

7. Resolve the vector $\mathbf{u} = (3,1,4)$ into components parallel and perpendicular to the vector $\mathbf{v} = (2,2,-3)$.

8. Calculate the distance d between the point (2, 0, –3) and the plane $2x + 1y - 3z = 5$.

9. Prove using vector methods (without components) that an angle inscribed in a semicircle is a right angle. Hint: Prove that $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = 0$.

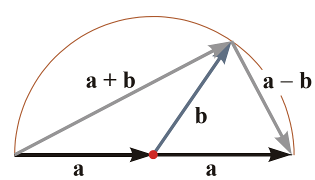


Figure 1.19

1.3 Cross Product

If **u** and **v** are two vectors then their cross product is denoted as $\mathbf{u}\times\mathbf{v}$ and is defined as a vector with these properties:

- The direction of $\mathbf{u}\times\mathbf{v}$ is perpendicular to both **u** and **v**. Since this still allows two possible directions for $\mathbf{u}\times\mathbf{v}$ we define the direction to be given by the right-hand rule. That is, suppose that we curl the fingers of our right hand from **u** toward **v** (by the smallest possible rotation). Then our thumb points in the direction of $\mathbf{u}\times\mathbf{v}$.
- The magnitude or length of $\mathbf{u}\times\mathbf{v}$ is given by the formula

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta\,. \tag{1.18}$$

Fig. 1.20 shows $\mathbf{u}\times\mathbf{v}$ for various orientations of **u** and **v**.

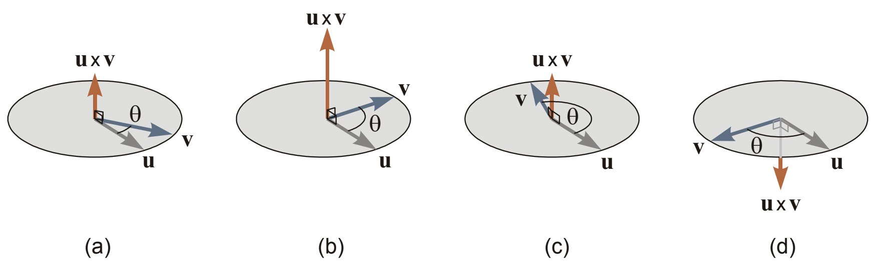


Figure 1.20 Illustration of the magnitude and direction of the cross product. In (a), (b) and (c) the shortest rotation causes the cross product to point upward; in (d) the shortest rotation causes it to point downward. When **u** and **v** are about 90° apart ((b) and (d)) the cross product is near its largest possible magnitude; When **u** and **v** are about 0° or 180° apart ((a) and (c)) the cross product is near its smallest possible magnitude.

If $\mathbf{u}=(u_1,u_2,u_3)$ and $\mathbf{v}=(v_1,v_2,v_3)$ then their cross product $\mathbf{u}\times\mathbf{v}$ can be computed from the following determinant formula.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{\mathbf{i}}\,(u_2v_3-u_3v_2)+\hat{\mathbf{j}}\,(u_3v_1-u_1v_3)+\hat{\mathbf{k}}\,(u_1v_2-u_2v_1) \tag{1.19}$$

We will accept (1.19) without proof. Some of the properties of the cross product are:

- For any vector **v**,

$$\mathbf{v}\times\mathbf{v}=\mathbf{0}=(0,0,0)\,. \tag{1.20}$$

- The cross product of the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$:

$$\hat{\mathbf{i}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}},\;\; \hat{\mathbf{j}}\times\hat{\mathbf{k}}=\hat{\mathbf{i}},\;\; \hat{\mathbf{k}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}} \tag{1.21}$$

- The **anti-**commutative property. For any vectors **u** and **v**,

$$\mathbf{u}\times\mathbf{v}=-\mathbf{v}\times\mathbf{u} \tag{1.22}$$

- The distributive property.

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \qquad (1.23)$$

An important application of the cross product is the construction of a vector that is perpendicular to two other vectors. For example when a charged particle enters a magnetic field[1] as shown in Fig. 1.21 it experiences a force, called the Lorentz force, whose direction is perpendicular to its direction of motion and to the direction of the magnetic field. If q is the particle's charge, $\mathbf{v}$ is the particle's velocity and $\mathbf{B}$ is the magnetic field, then the force, $\mathbf{F}$, on the particle is given by

$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B}\,. \qquad (1.24)$$

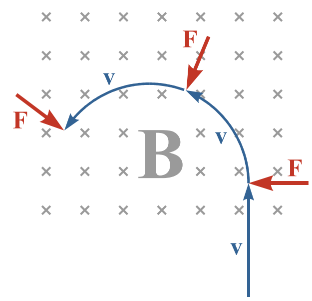


Figure 1.21 The motion of a charged particle in a magnetic field.

Since this force applies at every point along its motion, the charged particle is deflected and moves in a circular path. (Note that the x's in Fig. 1.21 represent a magnetic field directed into the page.[2])

Another application of the cross product is in the description of torque (or moment of force or rotational moment). The torque, $\boldsymbol{\tau}$, about a point, P, is defined by the formula

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}\,, \qquad (1.25)$$

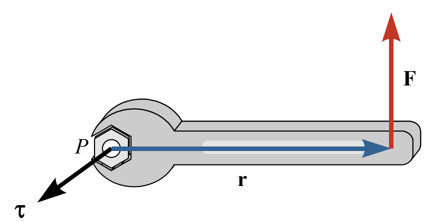


Figure 1.22 The torque or moment of force.

where $\mathbf{F}$ is the applied force and $\mathbf{r}$ is the radial vector pointing from point P to the point where the force is applied. $\boldsymbol{\tau}$ plays the same role in rotational motion that the force $\mathbf{F}$ plays in linear motion.

Example 1.9: Find the equation of the plane passing through the three points $\mathbf{P} = (1, 2, 3)$, $\mathbf{Q} = (-2, 0, 4)$ and $\mathbf{R} = (5, 2, -1)$.

Solution: Construct vectors $\overrightarrow{\mathbf{PQ}}$ and $\overrightarrow{\mathbf{PR}}$ and take their cross product to produce a vector $\mathbf{n}$ that is normal to the plane:

$$\overrightarrow{\mathbf{PQ}} = (-3, -2,\ 1)$$

$$\overrightarrow{\mathbf{PR}} = (4,\ 0, -4)$$

$$\mathbf{n} = \overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & -2 & 1 \\ 4 & 0 & -4 \end{vmatrix} = (8, -8,\ 8)$$

---

[1] We will study magnetic fields and other vector fields in Chapter 4.
[2] A vector pointing straight into the page is denoted $\times$ and a vector pointing straight out of the page is denoted $\odot$.

Now we can find the equation of the plane in the usual way (see Example 1.5):
If $\mathbf{r} = (x, y, z)$ denotes a general point in the plane and $\mathbf{P}$ is a point known to be on the plane, then $(\mathbf{r} - \mathbf{P}) \cdot \mathbf{n} = 0$ is the equation of the plane. In terms of components this is

$$\big((x, y, z) - (1,2,3)\big) \cdot (8, -8, 8) = 0,$$

or

$$8x - 8y + 8z = 16,$$

or

$$x - y + z = 2.$$

Example 1.10: Find the distance between the point $\mathbf{P} = (1, 3, 6)$ and the line defined parametrically by the formula $(x, y, z) = (1,2,4) + t(-1, -2, -3)$.

Solution: Recall that in Example 1.8 we found the distance between a point and a plane. That depended on finding a vector normal (i.e. perpendicular) to the plane because that vector defined the direction that gave the shortest distance. This problem is more difficult because there is no unique direction normal to a line.

Referring to Fig. 1.23 we will do the following:

1) Pick arbitrary points **Q** and **S** on the line.

Construct vectors $\overrightarrow{\mathbf{PS}}$ and $\overrightarrow{\mathbf{PQ}}$ and take their cross product to produce a vector **v** that is normal to Plane 1.

2) Take the cross product of vectors $\overrightarrow{\mathbf{QS}}$ and **v** to produce a vector **n** that is normal to Plane 2.

3) Construct the unit vector $\hat{\mathbf{n}} = \mathbf{n} / |\mathbf{n}|$ and take its dot product with the vector $\overrightarrow{\mathbf{PS}}$. Except for a possible minus sign this is the distance d.

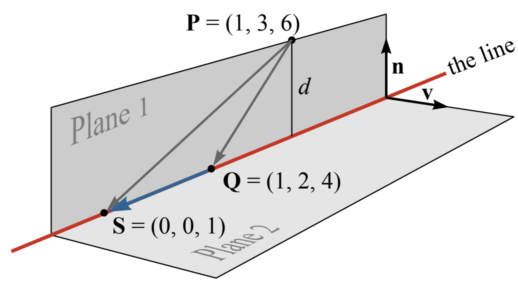


Figure 1.23 Finding the distance between a point **P** and a line.

Step 1: Letting $t = 0$ in the equation of the line gives $\mathbf{Q} = (1,2,4)$ and letting $t = 1$ gives $\mathbf{S} = (0, 0, 1)$.
Thus $\overrightarrow{\mathbf{PS}} = (-1,-3,-5)$ and $\overrightarrow{\mathbf{PQ}} = (0,-1,-2)$, and their cross product is

$$\mathbf{v} = \overrightarrow{\mathbf{PS}} \times \overrightarrow{\mathbf{PQ}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & -3 & -5 \\ 0 & -1 & -2 \end{vmatrix} = (1, -2,\ 1)$$

Step 2: Construct $\overrightarrow{\mathbf{QS}} = (-1,-2,-3)$ and find its cross product with **v**.

$$\mathbf{n} = \overrightarrow{\mathbf{QS}} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & -2 & -3 \\ 1 & -2 & 1 \end{vmatrix} = (-8, -2,\ 4)$$

Step 3: Construct the unit vector $\hat{\mathbf{n}} = \dfrac{\mathbf{n}}{|\mathbf{n}|} = \frac{1}{\sqrt{84}}(-8,-2,4)$ and take its dot product with $\vec{\mathbf{PS}}$.

$$\hat{\mathbf{n}} \cdot \vec{\mathbf{PS}} = \tfrac{1}{\sqrt{84}}(-8,-2,4)\cdot(-1,-3,-5) = \tfrac{1}{\sqrt{84}}(-6) = -0.655$$

Dropping the minus sign we find that the distance is $d = 0.655$.

Note: Another way to solve this problem which avoids the use of the cross product is letting $\mathbf{R} = (x,y,z) = (1,2,4) + t(-1,-2,-3)$ be a point on the line and solving the condition $\vec{\mathbf{PR}} \cdot \vec{\mathbf{QS}} = 0$ for t. (You will find that $t = -\frac{4}{7}$.) This value of t gives the point on the line that is closest to **P** and the distance can then be found using Pythagoras' theorem.

⮘

Problem Set 1.3 - Cross Product

1. Calculate $\mathbf{u}\times\mathbf{v}$ if $\mathbf{u} = (4,0,0)$ and $\mathbf{v} = (0,2,0)$.

2. Calculate $\mathbf{u}\times\mathbf{v}$ if $\mathbf{u} = (4,2,8)$ and $\mathbf{v} = (-3,2,2)$.

3. Calculate $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$ if $\mathbf{u} = (1,2,3)$ and $\mathbf{v} = (-3,2,0)$ and verify that they are the negative of each other.

4. Verify that $\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}$ and that $4\,\hat{\mathbf{i}}\times 2\,\hat{\mathbf{j}} = 8\,\hat{\mathbf{k}}$. Compare your answer with that of #1.

5. Find the equation of the plane containing the three points **P** = (1, 2, 3), **Q** = (–2, 2, 2) and **R** = (0, 0, –1).

6. Find the equation of the plane whose normal is perpendicular to the vectors **u** = (–2, 3, 0) and **v** = (0, 1, –1) and which contains the point **P** = (1, 2, 3).

7. Find the distance between the point **P** = (3, 3, 5) and the line defined parametrically by the formula $(x,y,z) = (4,0,0) + t(-1,-2,-3)$.

8. Find the area of the triangle having vertices at $P = (2, 0, 1)$, $Q = (3, 1, 0)$, and $R = (-1, 1, -1)$. Hint: Look at Fig. 1.14. The area of that triangle is $\frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta$ but according to Eq. (1.18) this is also equal to $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$.

1.4 Curves in Space

Representations of Curves. Curves can be represented in several ways:

- **As the intersection of two surfaces** $F_1(x, y, z) = 0$ and $F_2(x, y, z) = 0$. Fig. 1.3(c) showed an example where one surface was a cylinder $F_1(x, y, z) = x^2 + z^2 - 2^2 = 0$ and the other surface was a plane $F_2(x, y, z) = x - y + 4 = 0$. The intersection was an ellipse. This is not a convenient way to represent curves.

- **Parametrically**, as the set of three functions, $\{x = x(t),\ y = y(t),\ z = z(t)\}$, where the parameter *t* is defined over some interval $\alpha \le t \le \beta$. Substituting different values of *t* into the functions $x(t), y(t)$ and $z(t)$ results in different points (x, y, z) along the curve.

- As the set of points traced out by the head of the **position vector r**, where

$$\mathbf{r} \;=\; \mathbf{r}(t) \;=\; x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}} \;=\; \big(x(t), y(t), z(t)\big), \tag{1.26}$$

  as the parameter *t* varies over the interval $\alpha \le t \le \beta$. Substituting different values of *t* into the functions $x(t), y(t)$ and $z(t)$ results in different vectors **r** pointing to different points (x, y, z) along the curve. In later applications the parameter *t* will represent time and the vector **r** will represent the position of an object in space as it moves along the curve. But for now the parameter *t* is merely a parameter that allows us to describe different points along the curve.

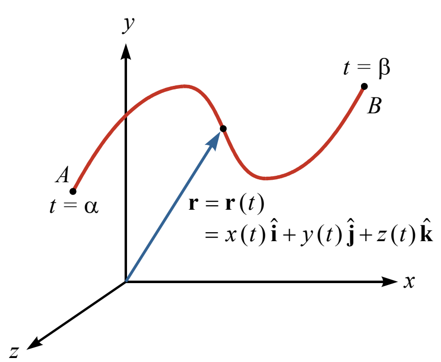


Figure 1.24 The position vector representation of a curve. As the parameter *t* varies, the vector **r** points to different points along the curve. When $t = \alpha$ the vector **r** points to the beginning of the curve at point *A* and when $t = \beta$ it points to the end of the curve at point *B*.

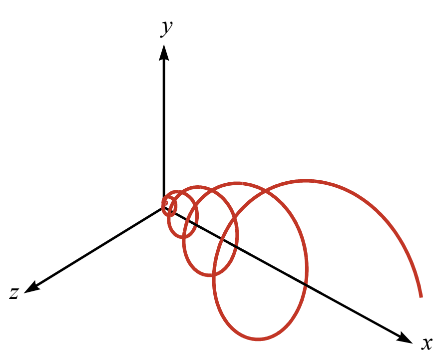


Figure 1.25 An example of the position vector representation of a curve. The formula

$$\mathbf{r}(t) = t\,\hat{\mathbf{i}} + t\sin t\,\hat{\mathbf{j}} + t\cos t\,\hat{\mathbf{k}}\,,$$

with parameter *t* in the interval $0 \le t \le 7\pi$ defines a growing spiral curve. The curve begins at the origin when *t* = 0 and ends at the point $(7\pi, 0, -7\pi)$ when $t = 7\pi$.

The Tangent Vector of a Curve. Just as $\mathbf{r}(t)$ is a vector giving the position of the curve at any parameter value t, we wish to construct a vector $\mathbf{T}(t)$ which gives the tangent of the curve at any parameter value t.

Refer to Fig. 1.26. Let $\mathbf{r}(t)$ be the position vector for parameter value t and $\mathbf{r}(t+h)$ be the position vector for parameter value $t+h$. It is clear that the difference vector $\mathbf{r}(t+h)-\mathbf{r}(t)$ is an approximation to the tangent at P. As h approaches zero this approximation becomes exact but the length of the vector becomes zero. To avoid this we divide by h and define the tangent vector $\mathbf{T}(t)$ as

$$\mathbf{T}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} .$$

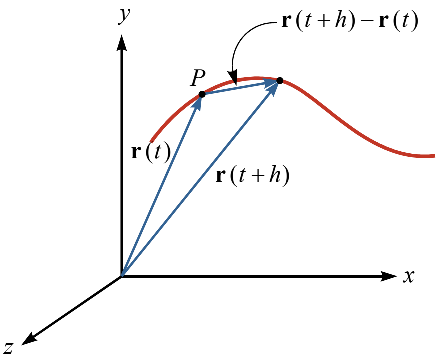


Figure 1.26 Construction of the tangent vector **T**.

We must check to make sure that this limit exists. Expanding the numerator gives

$$\mathbf{T}(t) = \lim_{h \to 0} \frac{\left[x(t+h)\,\hat{\mathbf{i}} + y(t+h)\,\hat{\mathbf{j}} + z(t+h)\,\hat{\mathbf{k}}\right] - \left[x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}\right]}{h}$$

or rearranging

$$\mathbf{T}(t) = \lim_{h \to 0} \frac{x(t+h)-x(t)}{h}\hat{\mathbf{i}} + \lim_{h \to 0} \frac{y(t+h)-y(t)}{h}\hat{\mathbf{j}} + \lim_{h \to 0} \frac{z(t+h)-z(t)}{h}\hat{\mathbf{k}}$$

Using the definition of the derivatives of functions x, y and z gives this formula for the **tangent vector** to the curve:

$$\mathbf{T}(t) = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}. \qquad (1.27)$$

The required limit exists, and therefore the tangent vector exists, as long as dx/dt, dy/dt and dz/dt exist.

Summary: For any vector function

$$\mathbf{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}},$$

we define its derivative as

$$\frac{d\mathbf{r}}{dt} = \lim_{h \to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} . \qquad (1.28)$$

If $\mathbf{r}$ represents the position vector of a curve then the derivative of $\mathbf{r}$ is a **tangent vector** to the curve

$$\mathbf{T}(t) = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} \qquad (1.29)$$

We also define the **unit tangent vector** as $\hat{\mathbf{T}} = \dfrac{\mathbf{T}}{|\mathbf{T}|}$. Note that $\mathbf{T}$ and $\hat{\mathbf{T}}$ point in the forward direction along the curve (i.e. in the direction of increasing parameter value t).

The Normal Vectors of a Curve. In the previous section we saw that at every point along a curve there exists a unit vector $\hat{\mathbf{T}}$ which is tangent to the curve at that point. In this section we define two more unit vectors, the **principal normal**, $\hat{\mathbf{N}}$, and the **binormal**, $\hat{\mathbf{B}}$, which are perpendicular to $\hat{\mathbf{T}}$ and to each other.

Fig. 1.27 shows an example of these three vectors at two points along a curve. At point A the curve lies in plane 1 and at point B it lies in plane 2. $\hat{\mathbf{N}}$ is the normal that points "*into*" the center of the curve so that $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ are both in the plane of the curve and $\hat{\mathbf{B}}$ is the normal that is normal to the plane of the curve.

First we construct $\hat{\mathbf{N}}$. Fig. 1.28 shows the unit tangent vector $\hat{\mathbf{T}}$ at two successive points along the curve. The vector triangle in the inset below the curve shows that the difference between the two unit tangent vectors is approximately normal to the curve. As h approaches zero this approximation becomes exact but the length of the vector becomes zero. To avoid this we divide by h and define the normal vector $\mathbf{N}(t)$ as

$$\mathbf{N}(t) = \lim_{h \to 0} \frac{\hat{\mathbf{T}}(t+h) - \hat{\mathbf{T}}(t)}{h} = \frac{d\hat{\mathbf{T}}}{dt} \tag{1.30}$$

Dividing $\mathbf{N}$ by its own magnitude gives the **principal normal vector** $\hat{\mathbf{N}}$:

$$\hat{\mathbf{N}} = \frac{\mathbf{N}}{|\mathbf{N}|} \tag{1.31}$$

To construct the **binormal** $\hat{\mathbf{B}}$ we simply take the cross product of $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$.

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} \tag{1.32}$$

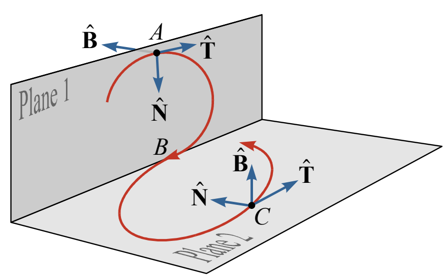


Figure 1.27 The triad of mutually perpendicular unit vectors: the unit tangent vector, $\hat{\mathbf{T}}$, the principal normal, $\hat{\mathbf{N}}$, and the binormal, $\hat{\mathbf{B}}$, shown at two points along a curve.

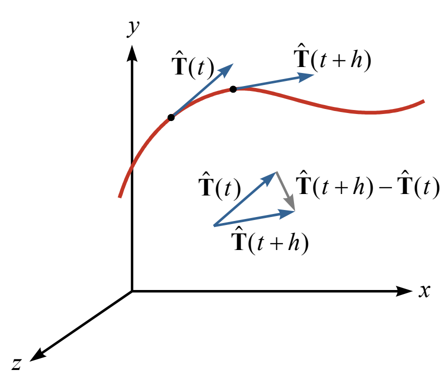


Figure 1.28 Constructing $\hat{\mathbf{N}}$.

Example 1.11: For the curve $\mathbf{r}(t) = t\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + t^2\,\hat{\mathbf{k}}$ find the vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ at the point P = (2, 2, 4).

Solution: First differentiate **r** to get **T** and then normalize to get $\hat{\mathbf{T}}$ as a function of *t*:

$$\mathbf{T}(t) = \frac{d\mathbf{r}}{dt} = 1\,\hat{\mathbf{i}} + 1\,\hat{\mathbf{j}} + 2\,t\,\hat{\mathbf{k}} \quad \rightarrow \quad \hat{\mathbf{T}}(t) = \frac{1\,\hat{\mathbf{i}} + 1\,\hat{\mathbf{j}} + 2\,t\,\hat{\mathbf{k}}}{\sqrt{2 + 4\,t^2}}$$

Now differentiate $\hat{\mathbf{T}}$ to find **N** (this takes some work because of the quotient rule!):

$$\mathbf{N}(t) = \frac{d\,\hat{\mathbf{T}}}{dt} = \frac{-4\,t\,\hat{\mathbf{i}} - 4\,t\,\hat{\mathbf{j}} + 4\,\hat{\mathbf{k}}}{\left(2 + 4\,t^2\right)^{3/2}}$$

These are all the derivatives that are required. Now we evaluate $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$ at point *P*. The parameter at point *P* is $t = 2$. We find that $\hat{\mathbf{T}}(2) = (0.24, 0.24, 0.94)$ and $\hat{\mathbf{N}}(2) = (-0.67, -0.67, 0.33)$. Taking the cross product of these gives $\hat{\mathbf{B}} = (0.707, -0.707, 0)$.

In practice finding **N** directly by calculating the derivative $\frac{d\,\hat{\mathbf{T}}}{dt}$ is usually quite difficult because the quotient rule is required. An alternative approach is finding **N** *indirectly* by calculating the easier derivative, $\frac{d\mathbf{T}}{dt}$. Unfortunately this vector has a component in the $\hat{\mathbf{T}}$ direction as well as a component in the $\hat{\mathbf{N}}$ direction as we can see if we write the vector **T** as $\mathbf{T} = |\mathbf{T}|\,\hat{\mathbf{T}}$ and then use the product rule to differentiate.

$$\frac{d\mathbf{T}}{dt} = \frac{d|\mathbf{T}|}{dt}\hat{\mathbf{T}} + |\mathbf{T}|\frac{d\,\hat{\mathbf{T}}}{dt} \tag{1.33}$$

$$= \frac{d|\mathbf{T}|}{dt}\hat{\mathbf{T}} + |\mathbf{T}|\,\mathbf{N}$$

$$= \alpha\,\hat{\mathbf{T}} + \beta\,\hat{\mathbf{N}} \tag{1.34}$$

But this is not a problem because we can then find the *direction* of **B** by taking the cross product of **T** with $\frac{d\mathbf{T}}{dt}$ and then find the *direction* of **N** by taking the cross product of **B** with **T**. Finally we normalize all three vectors to get the corresponding unit vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$. The reason this procedure works is that the cross product is distributive .

$$\mathbf{T} \times \frac{d\mathbf{T}}{dt} = \mathbf{T} \times \left(\alpha\,\hat{\mathbf{T}} + \beta\,\hat{\mathbf{N}}\right) = \alpha\,\underbrace{\mathbf{T} \times \hat{\mathbf{T}}}_{=0} + \beta\,\mathbf{T} \times \hat{\mathbf{N}} = \beta\,\mathbf{T} \times \hat{\mathbf{N}} = k\,\hat{\mathbf{B}} \tag{1.35}$$

Here is the procedure in point form:

Practical procedure to find $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$:

- Differentiate $\mathbf{r}(t)$ once to get $\mathbf{T}(t)$ and again to get $\dfrac{d\mathbf{T}}{dt}$.
- Calculate $\mathbf{T}\times\dfrac{d\mathbf{T}}{dt}=\mathbf{B}^*$. (The * means the direction is right but the length is not 1.)
- Calculate $\mathbf{B}^*\times\mathbf{T}=\mathbf{N}^*$.
- Divide each of $\mathbf{T}$, $\mathbf{N}^*$ and $\mathbf{B}^*$ by its own length to get unit vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$.

Example 1.12: For the curve $\mathbf{r}(t)=\left(t,\,2t,\,t^3-t\right)$ find the vectors $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$.

Solution: Follow the above steps:

- Differentiating $\mathbf{r}$ once gives $\mathbf{T}=\left(1,\,2,\,3t^2-1\right)$ and once again gives $\dfrac{d\mathbf{T}}{dt}=(0,\,0,\,6t)$.
- Calculate $\mathbf{T}\times\dfrac{d\mathbf{T}}{dt}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 2 & 3t^2-1\\ 0 & 0 & 6t\end{vmatrix}=(12t,\,-6t,\,0)\underset{\text{it's OK to scale this}}{\longrightarrow}(2,-1,\,0)=\mathbf{B}^*$.
- Calculate $\mathbf{B}^*\times\mathbf{T}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 2 & -1 & 0\\ 1 & 2 & 3t^2-1\end{vmatrix}=\left(-3t^2+1,\,-6t^2+2,\,5\right)=\mathbf{N}^*$.
- Normalize each of $\mathbf{T}$, $\mathbf{N}^*$ and $\mathbf{B}^*$ to get $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$.

$$\hat{\mathbf{T}}=\frac{\left(1,\,2,\,3t^2-1\right)}{\sqrt{5+\left(3t^2-1\right)^2}},\qquad \hat{\mathbf{N}}=\frac{\left(-3t^2+1,\,-6t^2+2,\,5\right)}{\sqrt{5\left(-3t^2+1\right)^2+25}},\qquad \hat{\mathbf{B}}=\frac{(2,-1,\,0)}{\sqrt{5}}$$

Problem Set 1.4 - Curves in Space

1. Calculate $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ at parameter value $t=\pi$ for the spiral curve shown in Fig. 1.25.

2. Calculate $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ as functions of t for the curve given by $\mathbf{r}=(t+5,\,-5t^2,\,t-1)$.

3. Find $\hat{\mathbf{T}}$, and $\hat{\mathbf{N}}$ for the plane (2-dimensional) curve $y=x^3$ at the point (–1,–1).

4. Find $\hat{\mathbf{T}}$, and $\hat{\mathbf{N}}$ as functions of t for the plane (2-dimensional) curve $x=2t+3,\ y=5-t^2$.

5. Calculate $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ as functions of t for the curve $\mathbf{r}(t)=(\cos t,\,\sin t,\,t)$.

6. Calculate $\hat{\mathbf{T}}$, $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ as functions of t for the curve $\mathbf{r}(t)=(t,\,t^2,\,t^2)$.

1.5 Motion in Space

In the previous sections $\mathbf{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}}$ was a **position vector** whose tip pointed to different positions along a curve as the parameter t varied over the interval $\alpha \le t \le \beta$. In this section the parameter t will represent time and the tip of the vector $\mathbf{r}$ will represent the position of a particle at time t as it moves along a curve. We will now derive formulas for the velocity and acceleration of the particle.

The **velocity v** is defined as the rate of change of the position of the particle.

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} \tag{1.36}$$

Note that this is exactly the same as the definition of the tangent vector, $\mathbf{T}$ (see Eq. 1.29). The magnitude of the velocity is known as the **speed** of the particle. The **acceleration a** is defined as the rate of change of the velocity of the particle.

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} + \frac{d^2z}{dt^2}\hat{\mathbf{k}} \tag{1.37}$$

Note that this is exactly the same as the definition of the vector $\frac{d\mathbf{T}}{dt}$. Recall that $\frac{d\mathbf{T}}{dt}$ has a component $\alpha\hat{\mathbf{T}}$ tangential to the curve and a component $\beta\hat{\mathbf{N}}$ normal to the curve (see Eq. (1.34)). In exactly the same way we can resolve the acceleration into components tangential and normal to the motion.

$$\mathbf{a} = a_T\hat{\mathbf{T}} + a_N\hat{\mathbf{N}} \tag{1.38}$$

The coefficient a_T is called the tangential acceleration and the coefficient a_N is called the normal or perpendicular acceleration. The tangential acceleration is due to changes in speed while the normal acceleration is due to motion around a curve. In the case of circular motion it is called the centripetal acceleration. Figure 1.27 shows their relationship. The easiest way to calculate a_T is to recall that the component of a vector $\mathbf{a}$ parallel to a unit vector $\hat{\mathbf{T}}$ is given by $(\mathbf{a}\cdot\hat{\mathbf{T}})\hat{\mathbf{T}}$ (see Eq. (1.16)). This means that

$$a_T = \mathbf{a}\cdot\hat{\mathbf{T}} = \mathbf{a}\cdot\hat{\mathbf{v}}\,. \tag{1.39}$$

The easiest way to calculate a_N is to then use Pythagoras' theorem (see Fig. 1.29).

$$|\mathbf{a}|^2 = a_T{}^2 + a_N{}^2 \tag{1.40}$$

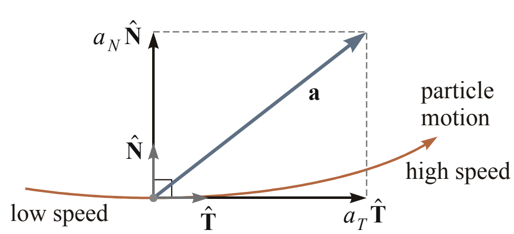


Figure 1.29 The components of the acceleration, **a**. a_T is the tangential acceleration and is due to changes in speed. a_N is the normal acceleration and is due to motion around a curve. Eqs. (1.39) and (1.40) state that $a_T = \mathbf{a}\cdot\hat{\mathbf{T}}$ and $a_N = \sqrt{|\mathbf{a}|^2 - a_T{}^2}$.

Example 1.13: A particle's position is given by $\mathbf{r}(t) = \left(t,\, t^2,\, t^3\right)$. Find the tangential and normal components of the acceleration at time $t = 1$.

Solution:

- Differentiating **r** twice gives the velocity $\mathbf{v} = \left(1,\, 2t,\, 3t^2\right)$ and the acceleration $\mathbf{a} = (0,\, 2,\, 6t)$.

  At $t = 1$ these become $\mathbf{v} = (1,\, 2,\, 3)$ and $\mathbf{a} = (0,\, 2,\, 6)$.
- This gives the unit tangent vector $\hat{\mathbf{T}} = \frac{1}{\sqrt{14}}(1,\, 2,\, 3)$ and the magnitude of the acceleration $|\mathbf{a}| = \sqrt{40}$.
- Calculate the tangential acceleration $a_T = \mathbf{a} \cdot \hat{\mathbf{T}} = \frac{1}{\sqrt{14}}(1,\, 2,\, 3) \cdot (0,\, 2,\, 6) = \frac{22}{\sqrt{14}} = 5.88$
- Calculate the normal acceleration $a_N = \sqrt{|\mathbf{a}|^2 - a_T{}^2} = \sqrt{\left(\sqrt{40}\right)^2 - \left(\frac{22}{\sqrt{14}}\right)^2} = 2.33$

Note: If the principal normal vector $\hat{\mathbf{N}}$ is also desired then we can calculate:

$$\mathbf{N} = \mathbf{a} - \left(\mathbf{a} \cdot \hat{\mathbf{T}}\right)\hat{\mathbf{T}} = (0,\, 2,\, 6) - \tfrac{22}{14}(1,\, 2,\, 3) = \tfrac{1}{7}(-11, -8,\, 9)$$

which means that

$$\hat{\mathbf{N}} = \tfrac{1}{\sqrt{266}}(-11, -8,\, 9)$$

Example 1.14 Uniform circular motion: Let $\mathbf{r} = R\cos(\omega t)\,\hat{\mathbf{i}} + R\sin(\omega t)\,\hat{\mathbf{j}}$ be the position vector of a particle, where R and ω are constants. Then this particle moves in a circle of radius R about the origin in the xy plane. The motion is counterclockwise with angular velocity ω radians per second. Show that:

- the speed of the particle is given by the well known formula $|\mathbf{v}| = R\,\omega$.
- the velocity vector **v** is tangent to the circle.
- the acceleration vector **a** and the position vector **r** are related by the equation $\mathbf{a} = -\omega^2\mathbf{r}$, which means that the acceleration is radially inward, and
- the magnitude of the acceleration is given by the well known formula $|\mathbf{a}| = \dfrac{|\mathbf{v}|^2}{R}$

Solution: The position vector is $\mathbf{r} = R\cos(\omega t)\,\hat{\mathbf{i}} + R\sin(\omega t)\,\hat{\mathbf{j}}$.

Differentiating **r** gives $\mathbf{v} = -R\,\omega\sin(\omega t)\,\hat{\mathbf{i}} + R\,\omega\cos(\omega t)\,\hat{\mathbf{j}}$.

Differentiating **v** gives $\mathbf{a} = -R\,\omega^2\cos(\omega t)\,\hat{\mathbf{i}} - R\,\omega^2\sin(\omega t)\,\hat{\mathbf{j}}$.

- The dot product of **v** with itself gives $\mathbf{v} \cdot \mathbf{v} = R^2\omega^2\left(\sin^2(\omega t) + \cos^2(\omega t)\right) = R^2\omega^2$. Taking the square root gives the speed $|\mathbf{v}| = R\,\omega$.
- The dot product of **v** with **r** gives $\mathbf{v} \cdot \mathbf{r} = 0$. Since **r** points radially outward from the center of the circle this means that **v** is tangent to the circle.

- Comparing the formulas for **a** and **r** shows that $\mathbf{a} = -\omega^2\mathbf{r}$.
- Substituting $\omega = |\mathbf{v}|/R$ into the formula $\mathbf{a} = -\omega^2\mathbf{r}$ gives $\mathbf{a} = -\mathbf{r}|\mathbf{v}|^2 / R^2$. Taking the magnitude of this formula gives $|\mathbf{a}| = \dfrac{|\mathbf{v}|^2}{R}$.

Uniform circular motion is extremely important because it occurs so often in physics and engineering and also because any more-complicated curve can be can be imagined to be made up of many arcs of various circles connected end to end.

Probably the most important result of this example is that as long as the speed v is constant then the acceleration has magnitude v^2 / R and points toward the center of the circle.

This argument can be reversed: if the acceleration is normal to the motion then the speed of a particle will remain constant. This is what happens in the case of a charged particle moving in a magnetic field as shown in Fig. 1.21 on p. 18.

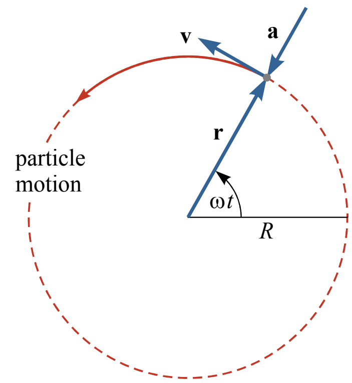


Figure 1.30 Uniform circular motion. The particle moves in a counterclockwise circle with constant angular velocity. At all points along the particle's path the position vector **r** is radially outward, the velocity vector **v** is tangential and the acceleration vector **a** is radially inward.

Problem Set 1.5 - Motion in Space

1. A particle's position is given by $\mathbf{r} = \left(\frac{1}{t}\cos t, \frac{1}{t}\sin t, -\frac{1}{10}t^2\right)$. Find the tangential and normal components of the acceleration at times $t = 2\pi$, 4π and 6π. Describe a situation in which a particle might have such motion.

2. For the space curve with position vector $\mathbf{r}(t) = e^t \cos t\,\hat{\mathbf{i}} + e^t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ find a_N and a_T at time $t = 0$.

3. Find the velocity and the acceleration as functions of time for a particle whose position is given by $\mathbf{r} = (t+5, -5t^2, t-1)$. Then find the speed and the tangential and normal components of the acceleration. Describe a situation in which a particle might have such motion. Notice that this curve is the same as the one given in #2 of Problem Set 1.4.

4. A particle moving along the curve $\mathbf{r} = (a\cos\omega t, b\sin\omega t, 0)$ traces out the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Show that, just as for the circle, the position and acceleration are related by $\mathbf{a} = -\omega^2\,\mathbf{r}$.

5. An object moving under the influence of gravity has its acceleration given by $\mathbf{a} = -g\,\hat{\mathbf{j}}$. Use two integrations to show that its position is given by $\mathbf{r} = \mathbf{r_o} + \mathbf{v_o}\,t - \frac{1}{2}g t^2\,\hat{\mathbf{j}}$, where $\mathbf{r_o}$ and $\mathbf{v_o}$ are (vector) constants of integration.

1.6 Arc Length

Recall Eq. (1.1) which stated that the distance d between any two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is given by Pythagoras' theorem,

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}\ . \tag{1.1}$$

If we connect these two points with a straight line then of course d will be the length of that line. Now consider a curve $\mathbf{r}(t) = (x(t), y(t), z(t))$ defined between $t = \alpha$ and $t = \beta$. We wish to derive a formula for the **total length** of this curve. To do this, imagine breaking the curve up into a large number, N, of short, straight line pieces as shown in Figure 1.31. (This could be done by evaluating $\mathbf{r}(t)$ at a number of equally spaced values of t between α and β and then connecting these points with straight line segments.)

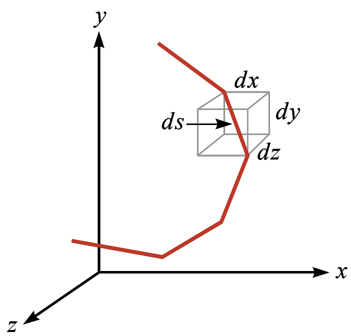


Figure 1.31 Calculating the length of a curve by replacing it by a large number of short, straight line segments connecting points along the curve. The segment shown has length given by Pythagoras' theorem,

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}\ .$$

If dx, dy and dz are the components in the x, y and z directions of one of the line segments (see Fig. 1.31) then the length of that line segment is

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}\ . \tag{1.41}$$

The length of the entire set of line segments is found by summing over all the segments.

$$L = \sum_{i=1}^{N} ds_i = \sum_{i=1}^{N} \sqrt{(dx_i)^2 + (dy_i)^2 + (dz_i)^2} \tag{1.42}$$

(In this formula ds_i is the length of the i^{th} line segment and dx_i, dy_i and dz_i are the dimensions of the i^{th} line segment.) If we take the limit as the number of segments goes to infinity this sum becomes an integral.

$$L = \int_A^B \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \tag{1.43}$$

(In this formula $\mathbf{A} = \mathbf{r}(\alpha)$ and $\mathbf{B} = \mathbf{r}(\beta)$ are the endpoints of the curve.) If we multiply and divide the integrand by the differential dt we get our final formula

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt\ . \tag{1.44}$$

In this formula α and β are the values of t at the endpoints of the curve and $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ are the derivatives of the components of **r**(t). The integral in (1.44) can be quite difficult to do.

Example 1.15: Calculate the length of the curve $\mathbf{r}(t) = \left(6t, 8t, t^2\right)$ between $t = 0$ and $t = 5$.

Solution: Differentiating **r** gives $\frac{d\mathbf{r}}{dt} = \left(6, 8, 2t\right)$. Substituting the components into (1.44) gives:

$$L = \int_0^5 \sqrt{6^2 + 8^2 + \left(2t\right)^2}\ dt = \int_0^5 \sqrt{100 + 4t^2}\ dt = 2\int_0^5 \sqrt{5^2 + t^2}\ dt$$

This integral can be done by using a trigonometric substitution or can be looked up in tables or can be done with the calculator (which uses Simpson's rule). The result is

$$L = 25\sqrt{2} - 25\ln\left(\sqrt{2} - 1\right) = 57.4$$

The **arc length function** $s(t)$ for a curve is defined as the length of the portion of the curve between $t = \alpha$ (the beginning of the curve) and $t = t$ (a variable point on the curve). Thus $s(\alpha) = 0$ and $s(\beta) = L$. The formula for the arc length function is

$$s\left(t\right) = \int_\alpha^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt\ . \qquad (1.45)$$

Example 1.16: Calculate the arc length function for the curve $\mathbf{r}(t) = (r\sin t, r\cos t, t)$, where r is a constant. This curve is a spiral with radius r and which rises in the z direction by 2π for each revolution of the spiral.

Solution: Differentiating **r** gives

$$\frac{dx}{dt} = r\cos t,\ \frac{dy}{dt} = -r\sin t \ \text{ and } \ \frac{dz}{dt} = 1.$$

Eq. (1.45) becomes

$$s\left(t\right) = \int_0^t \sqrt{\left(r\cos t\right)^2 + \left(-r\sin t\right)^2 + \left(1\right)^2}\ dt = \int_0^t \sqrt{r^2 + 1}\ dt = \underbrace{\sqrt{r^2 + 1}}_{\text{a constant}}\ t\ .$$

This answer could have been guessed by opening up the cylinder that the spiral lies on and laying it flat. The spiral would then be a diagonal straight line with a rise of 1 for a run of r.

Problem Set 1.6 - Arc Length

1. Find the arc length of the indicated portion of each curve.

(a) $\mathbf{r}(t) = \left(2\cos t,\, 2\sin t,\, \sqrt{5}\, t\right), \quad 0 \le t \le \pi$

(b) $x = t,\ y = 0,\ z = \frac{2}{3}t^{3/2}, \quad 0 \le t \le 8$

(c) $\mathbf{r}(t) = (t\sin t + \cos t)\,\hat{\mathbf{i}} + (t\cos t - \sin t)\,\hat{\mathbf{j}}, \quad \sqrt{2} \le t \le 2$

2. Write each of the following curves in parametric form and then find the arc length of the indicated portion. (Note: do the integrations numerically, i.e. use a calculator or computer.)

(a) $y = 5\ln(x),\ 1 \le x \le 10$

(b) $y = \sin(x), \quad 0 \le x \le \pi$

3. The position of a particle in terms of time is given by $\mathbf{r}(t) = 6t^3\,\hat{\mathbf{i}} - 2t^3\,\hat{\mathbf{j}} - 3t^3\,\hat{\mathbf{k}}, \quad 1 \le t \le 2$

(a) Find the arc length function $s(t)$.

(b) Parameterize the curve in terms of arc length, s.

(c) Find the position of the particle when t = 1sec., and when s = 1 m.

4. The curve in Fig. 1.32 is an exponential spiral $\mathbf{r}(t) = \left(e^{-t}\cos t,\, e^{-t}\sin t,\, 0\right)$. Consider a particle traveling along this curve with distances in meters.

a) Find the arc length function $s(t)$.

b) Find the total distance traveled, $0 \le t \le \infty$.

c) Find the position of the particle when it has traveled 1 meter.

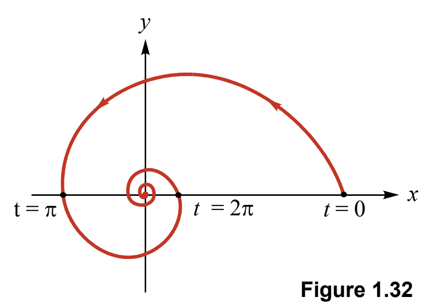


Figure 1.32

5. **Excel and Visual Basic Programming Problem.** There is no simple formula for the circumference of an ellipse. In this problem we will use Excel to calculate the circumference of the ellipse $\mathbf{r}(t) = (3\cos t, 2\sin t, 0)$ numerically. We will use two methods. (Note that in both methods we will actually calculate the arc length for one quarter of the ellipse and multiply the answer by 4 to get the total circumference.)

(a) Method 1. Using Eq. (1.43) Break up the quarter-ellipse into 20 line segments, find the position of the end of each segment, then the length of each segment and finally sum up the lengths. Here is an outline of the spreadsheet's formulas for the first few rows:

	A	B	C	D	E	F	G	H	I
	t	x	y	z	dx	dy	dz	ds	s(t)
5	0	=x(A5)	=y(A5)	=z(A5)					
6	0.0785	=x(A6)	=y(A6)	=z(A6)	=B6-B5	=C6-C5	=D6-D5	=SQRT(E6^2+F6^2+G6^2)	=H6+I5
7	0.1571	=x(A7)	=y(A7)	=z(A7)	=B7-B6	=C7-C6	=D7-D6	=SQRT(E7^2+F7^2+G7^2)	=H7+I6

Note that row 6 describes the first line segment, which starts at $t = 0$ and ends at $t = 0.0785$; row 7 describes the second segment, etc. Note that column I displays the running total for the ds's, namely the arc length function. Below are the values displayed in the first few rows.

t	x	y	z	dx	dy	dz	ds	s(t)
0	3	0	0					
0.0785	2.9908	0.1569	0	-0.0092	0.1569	0	0.1572	0.1572
0.1571	2.9631	0.3129	0	-0.0277	0.1560	0	0.1584	0.3156
0.2356	2.9171	0.4669	0	-0.0460	0.1540	0	0.1607	0.4763

(b) Method 2. Using Eq. (1.44) Open Excel's Visual Basic editor, insert a new module, and then enter the two functions SimpsonIntegral and Func listed below. SimpsonIntegral is a general purpose Simpson's rule integrator that integrates the function named Func from $t = A$ to $t = B$, using N strips. To find the circumference of the ellipse, simply type the formula =4 * SimpsonIntegral (0, Pi( ) / 2, 100) into any cell of the spreadsheet.

```
Public Function SimpsonIntegral (A, B, N)
  'Integrates a function called Func(x) between x=A and x=B using N strips.
  Dim X As Double, Dx As Double, Sum As Double, I As Integer

  N = (N \ 2) * 2                                    'Make sure N is an even number
  Dx = (B - A) / N                                   'Dx is the width of a strip
  Sum = 0

  For I = 0 To N
    X = A + I * Dx
    If I = 0 Or I = N Then
      Sum = Sum + Func(X)
    ElseIf I Mod 2 = 1 Then                          'I is odd
      Sum = Sum + 4 * Func(X)
    Else                                             'I is even
      Sum = Sum + 2 * Func(X)
    End If
  Next I

  SimpsonIntegral = Sum * Dx / 3
End Function

Public Function Func (X)
  'The integrand.
  Func = Sqr(9 * Sin(X) ^ 2 + 4 * Cos(X) ^ 2)
End Function
```

Answers, Chapter 1 – Vectors and Curves in Space

Problem Set 1.1 - Vectors, page 11

1. (a) 9.22
(b) $(-7,0,-6)$.
(c) $\frac{1}{\sqrt{85}}(-7,0,-6) = (-.76,\ 0,-.65)$

2.

(a)	5.10	(b)	12.4
(c)	(5, 3, 1)	(d)	(1, −1, 7)
(e)	(5, −1, 18)	(f)	(0.59, 0.20, 0.78)
(g)	(0.49, 0.49, −0.73)		

3. (8, 1, 0)
4. (−7, 5, 17)

5. (a) $\vec{PQ} = (1,-1,\ 1),\ \vec{QR} = (3,-3,\ 3) = 3\vec{PQ}$ $\rightarrow$ they are collinear.
(b) $\vec{PQ} = (-3,-4,-5),\ \vec{QR} = (9,11,13)$ $\rightarrow$ they are not collinear.

6. (b) $\frac{\mathbf{A}}{|\mathbf{A}|} = \frac{(-1,2,2)}{3}$ $\therefore\ \cos\alpha = -\frac{1}{3},\ \cos\beta = \frac{2}{3},\ \cos\gamma = \frac{2}{3}$
(c) $\alpha \cong 109.47°$, $\beta \cong 48.19°$, $\gamma \cong 48.19°$

Problem Set 1.2 - Dot Product, page 16

1. –4
2. –42
3. 136°
4. 154°
5. $1x + 2y + 1z = 4$
6. parallel component is (–.46, –.15, –.62); perpendicular component is (2.46, 2.15, –2.38)
7. parallel component is (–.47, –.47, .71); perpendicular component is (3.47, 1.47, 3.29)
8. 2.14

Problem Set 1.3 - Cross Product, page 20

1. (0, 0, 8)
2. (−12, −32, 14)
3. $\mathbf{u}\times\mathbf{v}$ = (–6, –9, 8) and $\mathbf{v}\times\mathbf{u}$ = (6, 9, –8)
5. $2x + 11y - 6z = 6$
6. $3x + 2y + 2z = 13$
7. 2.535
8. $\frac{1}{2}\left|\overline{PQ}\times\overline{PR}\right| = \frac{1}{2}\sqrt{42}$

Problem Set 1.4 - Curves in Space, page 25

1. $\hat{\mathbf{T}}$ = (0.290, –0.912, –0.290), $\hat{\mathbf{N}}$ = (–0.073, –0.324 0.943), $\hat{\mathbf{B}}$ = (–0.954, –0.253, –0.161)

2. $\hat{\mathbf{T}} = \dfrac{(1,-10t,1)}{\sqrt{100t^2+2}}$, $\hat{\mathbf{N}} = \dfrac{(-5t,-1,-5t)}{\sqrt{50t^2+1}}$, $\hat{\mathbf{B}} = \frac{1}{\sqrt{2}}(1,0,-1)$

3. $\hat{\mathbf{T}} = \dfrac{1}{\sqrt{10}}(1,3)$, $\hat{\mathbf{N}} = \dfrac{1}{\sqrt{10}}(3,-1)$

4. $\hat{\mathbf{T}} = \dfrac{(1,-t)}{\sqrt{1+t^2}}$, $\hat{\mathbf{N}} = \dfrac{(-t,-1)}{\sqrt{1+t^2}}$

5. $\hat{\mathbf{T}} = \dfrac{1}{\sqrt{2}}(-\sin t, \cos t, 1)$, $\hat{\mathbf{N}} = (-\cos t, -\sin t, 0)$, $\hat{\mathbf{B}} = \dfrac{1}{\sqrt{2}}(\sin t, -\cos t, 1)$

6. $\hat{\mathbf{T}} = \dfrac{(1,2t,2t)}{\sqrt{1+8t^2}}$, $\hat{\mathbf{N}} = \dfrac{(-4t,1,1)}{\sqrt{2+16t^2}}$, $\hat{\mathbf{B}} = \dfrac{1}{\sqrt{2}}(0,-1,1)$

Problem Set 1.5 - Motion in Space, page 28

1.

t	a_T	a_N
2π	.195	.165
4π	.1997	.080
6π	.1999	.053

This could describe an object in a fluid that is swirling down a drain in a vortex motion.

2. $a_N = \dfrac{\sqrt{6}}{3}$, $a_T = \dfrac{\sqrt{3}}{3}$

3. $\mathbf{v} = (1,-10t,1)$, $\mathbf{a} = (0,-10,0)$, $|\mathbf{v}| = \sqrt{100t^2+2}$, $a_T = \dfrac{100t}{\sqrt{100t^2+2}}$, $a_N = \dfrac{10\sqrt{2}}{\sqrt{100t^2+2}}$

This could describe an object with an initial velocity falling under the influence of gravity.

Problem Set 1.6 - Arc Length, page 31

1. (a) 3π
(b) $\frac{52}{3}$
(c) 1

2. (a) $x = t$ and $y = 5\ln(t)$. $L = 15.24$
(b) $x = t$ and $y = \sin(t)$. $L = 3.820$

3. (a) $s = 7t^3$
(b) $\mathbf{r}(s) = \left(\frac{6}{7}s, -\frac{2}{7}s, -\frac{3}{7}s\right)$
(c) $\mathbf{r}(t)\big|_{1\text{ sec}} = (6,-2,-3)$. $\mathbf{r}(s)\big|_{1\text{ m}} = \left(\frac{6}{7}, -\frac{2}{7}, -\frac{3}{7}\right)$

4. (a) $s = \sqrt{2}\left(1-e^{-t}\right)$
(b) $\sqrt{2}$ meters
(c) $\mathbf{r}(1\text{m}) \cong (0.0985, 0.2758, 0)$

5. 15.9