Example 5.14: The power p delivered by a generator to a circuit at any instant in time is given by the formula $p = iv$, where i and v are the current and voltage at that instant. Suppose that the generator in Fig. 5.30 produces an AC current $i = I_0 \sin(\omega t)$ and that the load causes the voltage to be out of phase with the current. To be precise, suppose that $v = V_0 \sin(\omega t + \varphi)$ (the phase shift φ could be positive or negative). Since i and v change with time, so does p. Find the *average* power $\langle p \rangle$ delivered by the generator over one complete cycle.

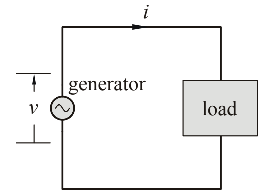

Figure 5.30 A capacitive or inductive load will cause a sinusoidal i and v to be out of phase.

Solution: We follow the same steps as in the previous example. Substituting in the formulas for i and v gives

$$p = i \cdot v = I_0 \sin(\omega t) \cdot V_0 \sin(\omega t + \varphi) = I_0 V_0 \cdot \underbrace{\sin(\omega t + \varphi) \cdot \sin(\omega t)}_{\text{apply Trig ID 11 to this}}. \tag{5.20}$$

We can change the product of trigonometric functions into a sum of trigonometric functions by using Trig ID 11 which says $2 \cdot \sin\alpha \cdot \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$. If we make the replacements $\alpha \to \omega t + \varphi$ and $\beta \to \omega t$ and divide both sides by 2 then Trig ID 11 reads

$$\sin(\omega t + \varphi) \cdot \sin(\omega t) = \tfrac{1}{2}\left[\cos(\varphi) - \cos(2\omega t + \varphi)\right].$$

Substituting this into (5.20) gives

$$p = I_0 V_0 \tfrac{1}{2}\left[\cos(\varphi) - \cos(2\omega t + \varphi)\right]$$

or expanding,

$$p = \underbrace{\frac{1}{2} I_0 V_0 \cos(\varphi)}_{\text{DC component}} - \underbrace{\frac{1}{2} I_0 V_0}_{\text{amplitude}} \cdot \cos(2\omega t + \varphi). \tag{5.21}$$

Because φ is a constant (remember it is the phase shift between v and i) this is again a sinusoidal waveform plus a DC component. The DC component gives the average power,

$$\langle p \rangle = \tfrac{1}{2} I_0 V_0 \cos\varphi. \tag{5.22}$$

__Some special loads__. **Resistors:** i and v are in phase ($\varphi = 0$) so (5.22) gives $\langle p \rangle = \tfrac{1}{2} I_0 V_0$. Since $V_0 = I_0 R$ this can also be written as $\langle p \rangle = \tfrac{1}{2} I_0^2 R$ which is the result we derived in the previous example. **Inductors:** $\varphi = +90^\circ$, **Capacitors** $\varphi = -90^\circ$. In both cases (5.22) gives $\langle p \rangle = 0$. The explanation is that inductors and capacitors absorb energy for one half of each cycle and then release the energy back to the generator in the other half of the cycle.

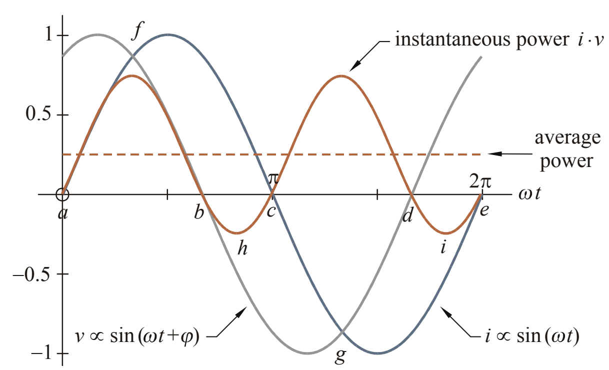

Figure 5.31 i and v are out of phase. The instantaneous power p, given by the product $i \cdot v$, is also sinusoidal but with a DC component. (This is the average power.) Note that at points h and i the power p is negative because i and v have opposite signs. Also note that if $\varphi = 0$ then this graph reduces to Fig. 5.29.