# 7. Complex Numbers

The study of complex numbers is important to us for many reasons. First, from an esthetic point of view, the real number system is incomplete. Many mathematical operations have no solution. Examples are $\arcsin(2)$, $\ln(-1)$, $\sqrt{-1}$, etc. But if we include complex numbers into our system then all these have solutions. The complex numbers complete the number system.

From a practical point of view, many real number problems have real solutions which cannot be found by using real arithmetic but can be found easily via complex arithmetic. The reason is that complex numbers add another dimension to the number system, thus opening up other routes to the solution. For example, it is almost impossible to analyze any AC circuit more complicated that a basic series *RLC* circuit without using complex numbers.

Finally complex numbers lead to remarkable insights. For example complex numbers show that there is a very close connection between sinusoidal oscillations and exponential decay. They help explain the different transients that can be observed in an *RLC* circuit when one changes *R*, *L* or *C*.

## 7.1 Imaginary Numbers

Technically the **square root** of a number $a$ is defined to be a number $x$ such that $x^2 = a$. For example 4 and –4 are both square roots of 16. The positive value, 4, is called the **principal value**[1]. The **radical symbol**, √ , refers *only* to the principal value. That is, $\sqrt{16} = 4$ , not –4, so

$$x^2 = 16 \text{ has two roots: } x = \sqrt{16} = 4 \text{ and } x = -\sqrt{16} = -4 .$$

Now we introduce imaginary numbers. Let's imagine that there is a number $j$ such that $j^2 = -1$ and let's see where this takes us. (It will be quite a ride!) Clearly $j$ is not a real number because no real number has this property. But whatever $j$ is, $-j$ will also have the property that squaring it gives –1. Let's call $j$ the principal value. We could keep using the radical symbol, like this:

$$x^2 = -1 \text{ has two roots: } x = \sqrt{-1} \text{ and } x = -\sqrt{-1} ,$$

but instead we use $j$:

$$x^2 = -1 \text{ has two roots: } x = j \text{ and } x = -j, \text{ where}$$

**definition:** $$j \equiv \sqrt{-1} . \qquad (7.1)$$

An **imaginary number** is defined as the square root of a negative number and $j$ is the unit of the imaginary numbers[2]. On the next page are some examples of arithmetic with imaginary numbers.

---

[1] Recall that arcsin, arccos and arctan also have a principal value.

[2] Your calculator and most mathematicians use $i$ as the imaginary unit but because $i$ is electric current we will use $j$.

**Example 7.1:** Some arithmetic with imaginary numbers.

- The square root of any negative number can be expressed as a multiple of $j$.

$$\sqrt{-9} = \sqrt{-1 \cdot 9} = \sqrt{-1} \cdot \sqrt{9} = j3 \quad \text{or} \quad 3j\,.$$

- This means that the equation $x^2 = -9$ has two roots, $x = 3j$ and $x = -3j$.

- Multiplying two imaginary numbers: $5j \cdot 8j = 40j^2 = -40$

- The powers of $j$ follow a pattern:

$$j^3 = (j^2)j = (-1)j = -j$$

$$j^4 = (j^2)(j^2) = (-1)(-1) = 1$$

$$j^5 = (j^4)j = (1)j = j$$

- Solving $x^2 - 6x + 13 = 0$ with the quadratic formula gives two roots that contain $j$.

$$x = \frac{6 \pm \sqrt{6^2 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4j}{2} = 3 \pm 2j$$

## 7.2 Complex Numbers

A **complex number** $z$ is the sum of a real number and an imaginary number. The last bullet in Example 7.1 shows that complex numbers appear naturally in the quadratic formula. A complex number can be written in the form

$$z = a + bj,$$

where $a$ and $b$ are both real numbers. $a$ is called the **real part** of $z$ and $b$ is called the **imaginary part** of $z$. We use the notation $a = \text{Re}\{z\}$ and $b = \text{Im}\{z\}$.

**Example 7.2:** If $z = -5 - 7j$ then $\text{Re}\{z\} = -5$ and $\text{Im}\{z\} = -7$. Note that $\text{Im}\{z\} = -7$ and not $-7j$.

Complex numbers can be represented on the **complex plane**. The real part of the complex number is plotted along the real (horizontal) axis and the imaginary part is plotted along the imaginary (vertical) axis. The real axis is just the original real number line.

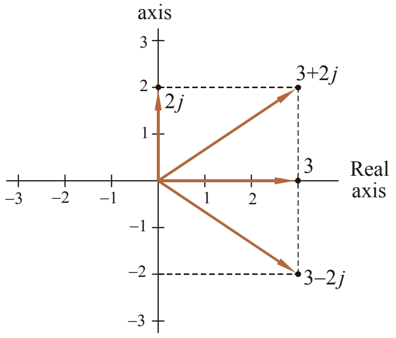


**Figure 7.1** The complex plane. Real numbers like 3 are on the real axis (originally the real number line). Imaginary numbers like 2$j$ are on the imaginary axis. Complex numbers like 3+2$j$ and 3–2$j$ are not on either axis.

Complex numbers are similar to 2-dimensional vectors in the way they are added, subtracted and stretched (scaled). But complex numbers can also be multiplied and divided whereas vectors can't. Here are some examples of arithmetic with complex numbers.

---

**Example 7.3:** Adding and subtracting complex numbers is the same as adding and subtracting vectors.

- $(1+2j)+(5-6j)=6-4j$
- $(1+2j)-(5-6j)=-4+8j$

---

**Example 7.4:** Multiplying or dividing a complex number by a real number is the same as stretching or shrinking a vector.

- $5\cdot(1+2j)=5+10j$
- $\dfrac{15+10j}{5}=\frac{1}{5}\left(15+10j\right)=3+2j$

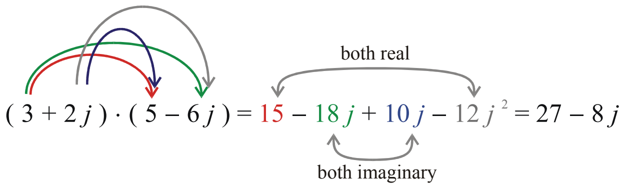
---

**Example 7.5:** Multiplying or dividing a complex number by an imaginary number not only stretches or shrinks but also rotates by $\pm 90^{\text{o}}$ in the complex plane. (More on this rotation later).

- $5j\cdot(1+2j)=5j+10j^2=-10+5j$
- $\dfrac{15+10j}{5j} \;=\; \dfrac{3}{j}+2 \;=\; 2-3j$
- $\dfrac{1}{j}=-j$ (you can verify this by cross-multiplying)

---

**Example 7.6:** Multiplying two complex numbers is done by expanding with the distributive law and simplifying. The 'first' and 'last' both contribute to the real part and the 'inner' and 'outer' both contribute to the imaginary part.



- $(3+2j)\cdot(5-6j)=15-18j+10j-12j^{2}=27-8j$

---

**Example 7.7:** To divide two complex numbers we multiply both numerator and denominator by the **complex conjugate** of the denominator. This leaves the denominator real. The complex conjugate of any complex number $z=a+bj$ is denoted $z^*$ and is defined as $z^*=a-bj$.

- $\dfrac{5+3j}{2-4j} \;=\; \dfrac{(5+3j)\;\cdot\;(2+4j)}{(2-4j)\;\cdot\underset{\substack{\text{complex conj.}\\ \text{of } 2-4j}}{\boxed{(2+4j)}}} \;=\; \dfrac{10+20j+6j+12j^2}{4+8j-8j-16j^2} \;=\; \dfrac{-2+26j}{20} \;=\; -0.1+1.3j$

---

## Problem Set 7.1 – Complex Numbers

In each question two numbers, $a$ and $b$, are given. Calculate $a+b$, $a-b$, $a \cdot b$, $a/b$, and $a^3$.

| | $a$: | $b$: |
|---|---|---|
| **1.** | $2+3j$ | $3-3j$ |
| **2.** | $2j$ | $4-3j$ |
| **3.** | $2j$ | $5j$ |
| **4.** | $3-2j$ | $1-4j$ |
| **5.** | $-1-1j$ | $1+1j$ |
| **6.** | $2+2j$ | $2-2j$ |
| **7.** | $x+yj$ | $x-yj$ |

# 7.3 Complex Numbers in Polar Coordinates

Complex numbers, like vectors, can be expressed in polar coordinates, using the notation $r\angle\theta$. Fig. 7.2 shows an example. The only problem with the polar notation is that there is no '$j$' in it so it is not clear whether $r\angle\theta$ refers to a complex number or a vector. (The exponential form of a complex number, which is introduced in section 7.4, remedies this).

The quantity $r$ in $r\angle\theta$ is called the **length** (or **magnitude** or **absolute value** or **modulus**) and the quantity $\theta$ is called the **angle** (or **argument**). For any complex number $z$, writing $|z|$ (i.e. using absolute value symbols) denotes its magnitude. For example the numbers $5$, $-5$, $3+4j$ and $5\angle 120^\circ$ are all a distance 5 from the origin so $|5|$, $|-5|$, $|3+4j|$ and $|5\angle 120^\circ|$ are all equal to 5.

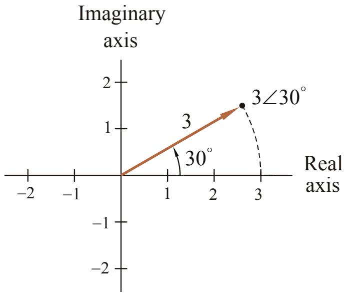


**Figure 7.2** Polar coordinates. The complex number shown is $3\angle 30^\circ$ (or $2.6+1.5j$ in rectangular).

Note that:
$r\angle 0^\circ$ is a positive real.
$r\angle 180^\circ$ is a negative real.
$r\angle 90^\circ$ is a positive imaginary.
$r\angle -90^\circ$ is a negative imaginary.

Complex numbers expressed in polar coordinates are difficult to add or subtract by hand (just like vectors in polar) but they are amazingly easy to multiply and divide. The rules are:

- **Multiplication rule:** To form the product multiply the magnitudes and add the angles.

$$(a\angle\theta)(b\angle\varphi) = (ab)\angle(\theta+\varphi) \tag{7.2}$$

- **Division rule:** To form the quotient divide the magnitudes and subtract the angles.

$$\frac{a\angle\theta}{b\angle\varphi} = \left(\frac{a}{b}\right)\angle(\theta-\varphi) \tag{7.3}$$

**Example 7.8:** Applying the multiplication and division rules (7.2) and (7.3).

- $(5\angle 30^{o})\cdot(3\angle 25^{o}) = 15\angle 55^{o}$
- $\dfrac{15\angle 32^{\circ}}{3\angle 25^{\circ}} = 5\angle 7^{\circ}$
- Proof that any number multiplied by $j$ is rotated 90° in the complex plane. (See Example 7.5.)

$$\left(r\angle\theta\right)\cdot j \;=\; \left(r\angle\theta\right)\cdot\left(1\angle 90^{\circ}\right) \;=\; r\angle\left(\theta+90^{\circ}\right)$$

- Example 7.7 redone by converting to polar, dividing, then going back to rectangular.

$$\frac{5+3j}{2-4j} \;=\; \frac{5.83\angle 0.540}{4.47\angle -1.107} \;=\; 1.30\angle 1.647 \;=\; -0.1+1.3j$$

$\prec$

---

**Proof of the multiplication rule.** The basic idea is to start with two complex numbers in polar, convert to rectangular, multiply, convert back to polar, interpret the result. Let's start with two numbers, $a\angle\theta$ and $b\angle\varphi$. Convert them to rectangular using trigonometry (sines and cosines).

$$\left(a\angle\theta\right)\left(b\angle\varphi\right) = \left(a\cos\theta + ja\sin\theta\right)\left(b\cos\varphi + jb\sin\varphi\right)$$

Expand the RHS using the distributive law. It becomes

$$= ab\cos\theta\cos\varphi + jab\cos\theta\sin\varphi + jab\sin\theta\cos\varphi + j^{2}ab\sin\theta\sin\varphi$$

The first and last terms are real. Group them together. They have a common factor *ab*. The other two terms are imaginary. Group them. They also have a common factor, *jab*.

$$= ab\left(\cos\theta\cos\varphi - \sin\theta\sin\varphi\right) + jab\left(\cos\theta\sin\varphi + \sin\theta\cos\varphi\right)$$

Use Trig IDs 6 and 7 to simplify the two expressions in the brackets.

$$= ab\cos\left(\theta+\varphi\right) + jab\sin\left(\theta+\varphi\right)$$

Convert this to polar using trigonometry and we have obtained formula (7.2).

$$= \left(ab\right)\angle\left(\theta+\varphi\right)$$

The proof of the division rule (7.3) is similar.

---

Your calculator is designed to do arithmetic with complex numbers. Instructions for using it were given previously on page 221 where we used complex mode to add and subtract vectors. The only additional instructions here are that we can also multiply and divide complex numbers (and even square and cube them.) **Warning**: As always, multiplication and division have precedence over addition and subtraction so if you want to multiply, say, $3+2j$ by $5-6j$ you **must use brackets** and enter it as $(3+2j)$ x $(5-6j)$.

## Problem Set 7.2 – Complex numbers in polar coordinates

**1.** Expressions such as the following occur in the analysis of AC circuits. Evaluate each expression. Give answers to 3 sig. figs. in both rectangular form and polar form.
Hint for (g): These angles can't be entered into the calculator so instead do the division 'by hand' using the division rule. To convert to rectangular use trigonometry.

**(a)** $\dfrac{(5+2j)(7-2j)}{(5+2j)+(7-2j)}$ **(b)** $\dfrac{1}{5+2j}+\dfrac{1}{4j}+\dfrac{1}{6-3j}$

**(c)** $\dfrac{1}{\dfrac{1}{5+3j}+\dfrac{1}{7+2j}}$ **(d)** $\dfrac{50\angle 30^\circ}{4\angle 5^\circ}-7+8j$

**(e)** $\dfrac{50\angle 30^\circ-(7+8j)}{4\angle 5^\circ}$ **(f)** $\dfrac{5\angle 60^\circ}{100\angle 15^\circ}\cdot 2\angle 20^\circ$

**(g)** $\dfrac{50\angle(\omega t+45^\circ)}{2.5\angle(\omega t-15^\circ)}$

**2.** Find the required real or imaginary part of the complex expression to 3 sig. figs.
Hint for (c) through (i): See the hint for # 1.

**(a)** $\operatorname{Re}\left\{\dfrac{(5+2j)(7-2j)}{(5+2j)+(7-2j)}\right\}$ **(b)** $\operatorname{Im}\left\{\dfrac{50\angle 30^\circ}{4\angle 5^\circ}-7+j8\right\}$

**(c)** $\operatorname{Im}\left\{\dfrac{50\angle(100t+45^\circ)}{25\angle(100t+35^\circ)}\right\}$ **(d)** $\operatorname{Re}\{5\angle(\omega t+45^\circ)\}$

**(e)** $\operatorname{Im}\{5\angle \omega t\}$ **(f)** $\operatorname{Im}\left\{\dfrac{50\angle \omega t}{3+4j}\right\}$

**(g)** $\operatorname{Re}\left\{\dfrac{1\angle(12t)}{-5+12j}\right\}$ **(h)** $\operatorname{Re}\left\{\dfrac{(e^{-5t})\angle(12t)}{-5+12j}\right\}$

**(i)** $\operatorname{Im}\left\{\dfrac{e^{5t}\angle(12t)}{5+12j}\right\}$

**3.** Solve for $v$: $\dfrac{5\angle 25^\circ}{3+4j}=\dfrac{v}{7-2j}$

**4.** Solve for $i$: $i+20\angle 30^\circ=\dfrac{100\angle 10^\circ}{5+2j}+2-3j$

**5.** Solve for $R$ and $L$, where $R$ and $L$ are both real numbers: $\dfrac{200\angle 86^\circ}{10\angle 26^\circ}=R+1000L\,j$

**6.** Solve for $R$ and $L$, where $R$ and $L$ are both real numbers: $\dfrac{10\angle 26^\circ}{200\angle 86^\circ}=\dfrac{1}{R}+\dfrac{1}{1000L\,j}$

**7.** Solve for $v$: $$\frac{\begin{vmatrix} 100\angle 0^\circ & -7-2j \\ v & 7-3j \end{vmatrix}}{\begin{vmatrix} 10+2j & -7-2j \\ -7-2j & 7-3j \end{vmatrix}} = 10\angle -60^\circ$$

---

Many trigonometric laws and identities are easy to prove with complex numbers and the multiplication rule (7.2). Among them are the sine and cosine laws, the sum of angles Trig IDs 6 and 7, and De Moivre's formula.

## Proof of the sine law and cosine law

This proof uses the **complex conjugate** $z^*$ which was introduced in Example 7.7. To recap:

In rectangular, if $z = a + b\,j$ then $z^* = a - b\,j$. In polar, if $z = r\angle\theta$ then $z^* = r\angle(-\theta)$

To get the complex conjugate just change every $j$ in an expression to $-j$ or change every $\theta$ to $-\theta$. The complex conjugate is useful because $z + z^* = 2a = 2\,r\cos\theta$ is always real, $z - z^* = 2\,b\,j = 2\,r\,j\sin\theta$ is always imaginary, and $z \cdot z^* = a^2 + b^2 = r^2$ is always real and positive.

Fig. 7.3 shows three complex numbers satisfying the relationship $b\angle\theta - a\angle 0 = c\angle\varphi$. Notice that $a$, $b$ and $c$ are also the lengths of the sides of the gray triangle. Each number has a complex conjugate which is shown in gray. ($a\angle 0$, being real, is its own complex conjugate.) They obey the relationship $b\angle(-\theta) - a\angle 0 = c\angle(-\varphi)$. If we multiply these two equations, expand and simplify, the cosine law pops out.

$$\big(b\angle\theta - a\angle 0\big)\cdot\big(b\angle(-\theta) - a\angle 0\big) = c\angle\varphi \cdot c\angle(-\varphi)$$

Expand the LHS:

$$b^2\angle 0 - ab\angle\theta - ab\angle(-\theta) + a^2\angle 0 = c^2\angle 0$$

Simplify:

$$b^2 - \underbrace{\big((ab)\angle\theta + (ab)\angle(-\theta)\big)}_{z\,+\,z^*\text{ is real, }=\,2ab\cos\theta} + a^2 = c^2$$

This is the **cosine law**, $c^2 = a^2 + b^2 - 2\,a\,b\cos\theta$. To prove the sine law just take the imaginary part of $b\angle\theta - a\angle 0 = c\angle\varphi$. This gives $b\sin\theta - 0 = c\sin\varphi$ or $\dfrac{\sin\varphi}{c} = \dfrac{\sin\theta}{b}$, which is the **sine law**.

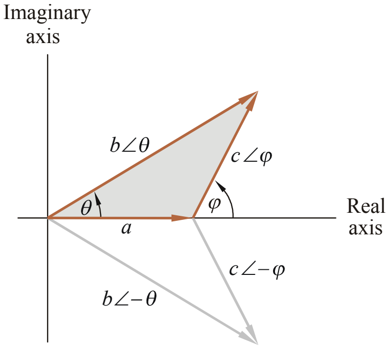


**Figure 7.3** Setup to prove the sine and cosine laws. The complex numbers obey $a\angle 0 + c\angle\varphi = b\angle\theta$.

## Proof of Trig ID 6: *sin (θ + φ) = sin θ · cos φ + cos θ · sin φ*

Start with the multiplication rule in the form $(1\angle\theta)\cdot(1\angle\varphi)=1\angle(\theta+\varphi)$. Convert each complex number to rectangular using trigonometry. (That is, use $r\angle\alpha = r\cos\alpha + jr\sin\alpha$ ).

$$(\cos\theta + j\sin\theta)\cdot(\cos\varphi + j\sin\varphi) = \cos(\theta+\varphi) + j\sin(\theta+\varphi)$$

Expand the LHS and simplify.

$$\underbrace{\cos\theta\cos\varphi - \sin\theta\sin\varphi}_{\text{real part}} + j\underbrace{(\cos\theta\sin\varphi + \sin\theta\cos\varphi)}_{\text{imaginary part}} = \underbrace{\cos(\theta+\varphi)}_{\text{real part}} + j\underbrace{(\sin(\theta+\varphi))}_{\text{imaginary part}}$$

Equating imaginary parts gives the sine identity. Equating real parts gives the cos identity.

## De Moivre's formula

This formula states that $(1\angle\theta)^n = 1\angle(n\theta)$, where $n$ is any integer. It can be generalized to read $(r\angle\theta)^n = r^n\angle(n\theta)$. To prove it just multiply $r\angle\theta$ by itself repeated and simplify. For example

$$(2\angle 40°)^3 = (2\angle 40°)(2\angle 40°)(2\angle 40°) = (2^3)\angle(40° + 40° + 40°) = 8\angle 120°$$

One application of De Moivre's formula is to prove the double, triple and higher multiple-angle identities. For example start with $(1\angle\theta)(1\angle\theta) = 1\angle(2\theta)$ and convert to rectangular.

$$(\cos\theta + j\sin\theta)\cdot(\cos\theta + j\sin\theta) = \cos(2\theta) + j\sin(2\theta)$$

Expand the LHS and simplify.

$$\underbrace{\cos^2\theta - \sin^2\theta}_{\text{real part}} + j\ \underbrace{2\sin\theta\cos\theta}_{\text{imaginary part}} = \underbrace{\cos(2\theta)}_{\text{real part}} + j\ \underbrace{\sin(2\theta)}_{\text{imaginary part}}$$

Equating real parts gives Trig ID 16 and equating imaginary parts gives Trig ID 15.

DeMoivre's formula can also be used to find the $n$ $n^{\text{th}}$ roots of any complex number $r\angle\theta$. One root (called the principal root) can be found by taking the $n^{\text{th}}$ root of the length and one-$n^{\text{th}}$ of the angle. Thus it is $r^{1/n}\angle(\theta/n)$. The other $n$ – 1 roots are distributed uniformly in a circle about the origin in the complex plane. This distribution guarantees that they all produce the same $n^{\text{th}}$ power.

For example let's find the cube roots of 8. Write 8 as $8\angle 0°$. The principal cube root is $8^{1/3}\angle\frac{0°}{3} = 2\angle 0°$. According to De Moivre's theorem all three roots are located uniformly in a circle about the origin as shown in Fig. 7.4. Let's verify that they are cube roots of 8 by cubing them:

$$(2\angle 0°)^3 = 8\angle 0° = 8$$

$$(2\angle 120°)^3 = 8\angle 360° = 8$$

$$(2\angle 240°)^3 = 8\angle 720° = 8$$

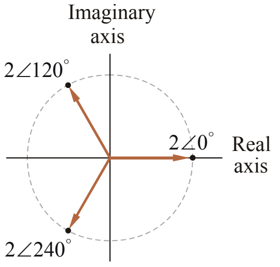


**Figure 7.4** The three complex cube roots of 8.

## 7.4 The Exponential Form and Euler's Formula

In Formulas (7.2) and (7.3) the adding or subtracting of angles when multiplying or dividing is reminiscent of the rules for exponents: $x^m \cdot x^n = x^{m+n}$ and $x^m / x^n = x^{m-n}$. They suggest that perhaps the angles are some kind of exponents. This guess turns out to be correct. The purpose of this section is to show that there is a third form in which we can write a complex number. It is called the **exponential form**. It is very similar to the polar form and at first glance seems to be just a notational change:

$$\underbrace{r\angle\theta}_{\text{polar form}} \equiv \underbrace{r\cdot e^{j\theta}}_{\text{exponential form}} \tag{7.4}$$

But $r\cdot e^{j\theta}$ *really is* an exponential function. Let us compare it to the exponential growth function given in chapter 4 on page 154. Then we see that *r* is the starting value. The base *e* is the same familiar number 2.718…. The variable $\theta$ plays the role of time. The unusual thing is the growth rate. It is imaginary since $j = \sqrt{-1}$ . To better understand (7.4) let $r = 2$ and let $\theta = 0, 1, 2, 3, \ldots$ Then (7.4) reads:

$$\begin{aligned} 2\cdot e^{j0} &= 2\angle 0 \\ 2\cdot e^{j1} &= 2\angle 1 \\ 2\cdot e^{j2} &= 2\angle 2 \\ &\vdots \end{aligned}$$

This is a sequence of complex numbers that traces out a circle of radius 2 about the origin of the complex plane. Compare this to the exponential growth function $y = 2\cdot e^{1t}$ (whose only difference is that the growth rate is real). Letting $t = 0, 1, 2, 3, \ldots$ gives an exponentially growing sequence of real numbers:

$$\begin{aligned} 2\cdot e^{0} &= 2 \\ 2\cdot e^{1} &= 5.4 \\ 2\cdot e^{2} &= 14.8 \\ &\vdots \end{aligned}$$

Although the polar and exponential forms both contain the same information (the same two variables, *r* and $\theta$) the exponential form has two advantages over the polar form. The first one (not that important) is that the exponential form *has* a '*j*' in it reminding us that it is a complex number. The second one (very important) is that all the properties of exponents that we learned in Chapter 4 apply to it. For example here is the same multiplication done in polar and exponential forms:

$$(15\angle 3)\cdot(3\angle 2) = 45\angle 5 \qquad \leftarrow \text{ polar form}$$

$$(15e^{3j})\cdot(3e^{2j}) = 45e^{5j} \qquad \leftarrow \text{ exponential form}$$

The polar form requires rule (7.2) while the exponential form only uses the properties of exponents. Now we will prove (7.4) but we will change its form a bit first. Convert the polar form on the left-hand-side into rectangular form using trigonometry. This gives

$$r\cos\theta + jr\sin\theta \equiv r\cdot e^{j\theta}$$

Now divide through by *r*. Then we have:

**Euler's formula** $$e^{j\theta} \equiv \cos\theta + j\sin\theta. \tag{7.5}$$

Richard Feynman called this the most remarkable formula in all of mathematics. Discovered by Euler in 1750, it shows a deep connection between exponential growth and sinusoidal oscillations. In essence it says that a system with a real time constant grows or decays exponentially and a system with an imaginary time constant oscillates. A special case of Euler's formula occurs when $\theta = \pi$. Then it reduces to $e^{j\pi} = -1$ or $e^{j\pi} + 1 = 0$. This is called **Euler's identity** and it shows an amazing connection between the numbers $e$, $\pi$, $j$ and 1!

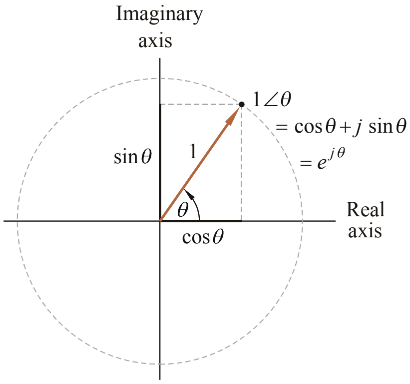


**Figure 7.5** The complex number $e^{j\theta}$ traces out a unit circle in the complex plane at $\theta$ increases.

The same number can also be written in rectangular form as $\cos\theta + j\sin\theta$ (this is Euler's formula) and in polar as $1\angle\theta$.

# Proof of Euler's Formula

Originally the sin, cos and tan functions were defined as ratios of the sides of a right triangle but in modern mathematics these and most other functions are defined by their **Taylor series**. The reason is that Taylor series are easy to work with and they provide a way for the calculator or computer to evaluate the functions to great precision. We will prove Euler's formula the same way he did – by writing down the Taylor series for the LHS of the formula and showing, after rearranging terms, that it equals the Taylor series for the RHS of the formula. (Note that we will have to assume that these Taylor series do in fact exist. We will derive them in the next book, *Calculus for Electrical Technology*. (See Chapter 9, page 211.)

The Taylor series for $e^x$, $\sin\theta$ and $\cos\theta$ are:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{7.6}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \tag{7.7}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \tag{7.8}$$

In these formulas ! is called **factorial** notation. 4! for example means $4 \cdot 3 \cdot 2 \cdot 1$. The dots … mean that the series goes on forever. If we keep only the first few terms then we get an approximation to the function. The more terms we keep, the better the approximation. For example $e^1$ or $e$ approximated by just the first 5 terms of (7.6) is

$$e^1 \approx 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} = 2.708333$$

This is already accurate to 0.4%. To prove Euler's formula we start by writing down the Taylor series for the LHS of Euler's formula. This is just (7.6) with $x$ replaced by $j\theta$.

$$e^{j\theta} = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots$$

Now we simplify. For example $(j\theta)^2 = -\theta^2$. Notice that the terms alternate; real, imaginary, … Now group all the real terms together and group all the imaginary terms together. The first group is the Taylor series for cos $\theta$ and the second group is the Taylor series for sin $\theta$ (multiplied by *j*).

$$e^{j\theta} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)}_{\cos\theta} + j\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin\theta}$$

$$= \cos\theta + j\sin\theta.$$

This equals the RHS of Euler's formula so the formula is proven.

# 7.5 Rotating Complex Numbers and Phasors

In Chapter 6 we considered a sinusoidal waveform to be the vertical component of a rotating vector and defined the phasor to be the rotating vector at the moment $t=0$. We now modify this definition.

**Second Definition:** A sinusoidal waveform is now considered to be the imaginary part of a *complex number* that is rotating counter-clockwise with constant angular velocity. A **phasor** is that *complex number* at the moment $t=0$.

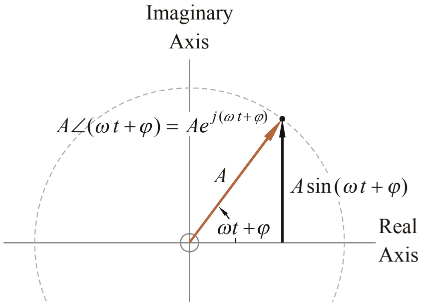


**Figure 7.6** The new definition of the phasor replaces the vector with a complex number.

This new definition has several advantages over the old one. The first advantage is that complex numbers can be divided whereas vectors can't. Consider the voltage and current graphed in Fig. 7.7 (left). Ohm's Law states that for resistors the ratio of voltage to current is a constant which we call the resistance, $R=v/i$. It is clear that in this case $R=2\Omega$ since

$$R = \frac{v}{i} = \frac{20\sin(2\pi f t)}{10\sin(2\pi f t)} = 2.$$

Now consider Fig. 7.7 (right) where the voltage and current are out of phase. We will see in Chapter 8 that inductors and capacitors will cause this to happen.

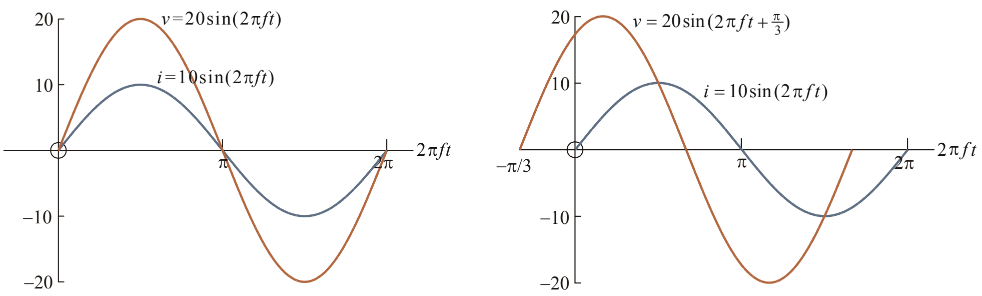


**Figure 7.7** (left) Waveforms *v* and *i* are in phase. Dividing *v* by *i* makes sense, and it gives the resistance. (right) *v* and *i* are out of phase. Dividing *v* by *i* is pointless. It gives no useful information.

Ohm’s Law is not valid since

$$\frac{v}{i} = \frac{20\sin(2\pi f t + \pi/3)}{10\sin(2\pi f t)} \neq \text{constant.}$$

But there are two important pieces of information contained in the graph:

- the ratio of the peak values (of the *amplitudes*) is $2\,\Omega$.
- the phase shift by which the voltage leads the current is $\pi/3$ radians.

Now notice what happens if we divide the voltage by the current, not as waveforms, but as complex numbers or as phasors:

$$\frac{v}{i} = \underset{\substack{\uparrow \\ \text{as rotating} \\ \text{complex numbers} \\ \text{in exponential form}}}{\frac{20\,e^{j(2\pi ft+\pi/3)}}{10\,e^{j2\pi ft}}} = \underset{\substack{\uparrow \\ \text{as rotating} \\ \text{complex numbers} \\ \text{in polar form}}}{\frac{20\angle(2\pi f t+\pi/3)}{10\angle(2\pi f t)}} = \underset{\substack{\uparrow \\ \text{as phasors} \\ \text{in polar form}}}{\frac{20\angle(\pi/3)}{10\angle 0}} = 2\angle\frac{\pi}{3} \equiv Z$$

The result contains the two pieces of information! We call this complex-valued constant the **impedance** and denote it by the letter $Z$. The impedance has a magnitude that gives the ratio of the amplitude of $v$ to that of $i$ and an angle that gives the phase shift by which $v$ leads $i$. Thus Ohm’s Law is saved by complex numbers and the complex impedance gives us two pieces of useful information about how $v$ and $i$ are related. AC circuits, no matter how complicated, can be analyzed with this new form of ‘Ohm’s Law’ and with the use of complex numbers.

The second advantage of defining phasors as complex numbers is that they have an exponential form that is easy to work with whereas vectors don’t. For example consider the exponentially decaying oscillation shown in Fig. 7.8. (We will see in the book *Differential Equations for Electrical Technology* that functions of this form describe voltage and current transients in *LRC* circuits.) It could be imagined to be *either* the $y$ component of a rotating vector *or* the imaginary component of a rotating complex number, both of which are expressed in polar coordinates as $(8e^{-0.33t})\angle(2t)$. (This means that the length decays with time as the angle increases with time.) If $(8e^{-0.33t})\angle(2t)$ is a *complex number* then we can also write it in exponential form as

$$8e^{-0.33t}\,e^{j2t} = 8e^{(-0.33+2j)t}.$$

Notice that we are able to combine the two exponents into one. The last form has many advantages from an algebra point of view because there is only one exponential and there are no sines or cosines. We can express the curve as

$$y = \text{Im}\,\{8e^{(-0.33+2j)t}\}.$$

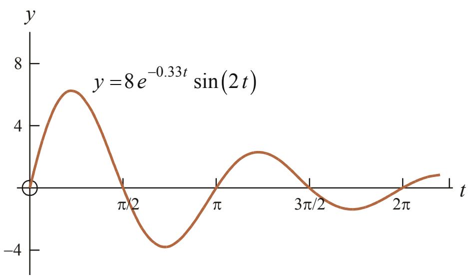


**Figure 7.8** This curve can be considered to be the imaginary part of the rotating, decaying complex number $8e^{(-0.33+2j)t}$.

## Problem Set 7.3 – Complex numbers in exponential form

**1.** Evaluate the following in rectangular, polar and exponential forms. Give answers to 3 sig. figs.

**(a)** $(3+5j)^5$

**(b)** $(5+j3)(7+j2)$

**(c)** $(7-j2)(-1+j7)$

**(d)** $(7-j2)+5\angle 75^{\text{o}}$

**(e)** $(2e^{j7})+(7e^{j2})$

**(f)** $(6e^{j5})(7e^{-j2})$

**(g)** $(6e^{j5})-(7e^{-j2})$

**(h)** $(6e^{j})+(5+j10)$

**(i)** $(5\angle 45^{\text{o}})(6e^{j\pi})$

**2.** Find the required real or imaginary part of each complex expression to 3 sig. figs.
Hint for (c): The exponent must be split apart like this: $e^{(-5+12j)t}=e^{-5t}\cdot e^{12jt}=(e^{-5t})\angle(12t)$.
As $t$ increases this traces out an inward spiral in the complex plane.

**(a)** $\operatorname{Im}\left\{\dfrac{5e^{2jt}}{6+8j}\right\}$

**(b)** $\operatorname{Re}\left\{\dfrac{e^{10jt}}{-5+12j}\right\}$

**(c)** $\operatorname{Re}\left\{\dfrac{e^{(-5+12j)t}}{-5+12j}\right\}$

**(d)** $\operatorname{Im}\left\{\dfrac{e^{(5+12j)t}}{5+12j}\right\}$

**3.** Find the roots of the complex equation $z^4 = 9.85 + 12.6j$ and show them in the complex plane.
Hint: They can be found with De Moivre's theorem.

**4.** Make a sketch of $e^{(1+12j)t}$ in the complex plane for $0 \le t \le 2$.

**5.** By adding and subtracting Euler's formula and its complex conjugate show that

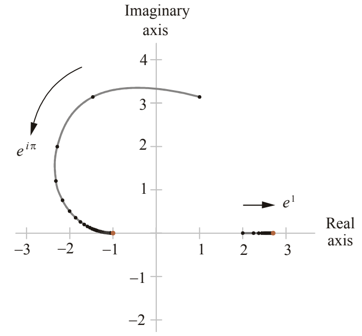
$$\cos\theta=\frac{e^{j\theta}+e^{-j\theta}}{2} \quad \text{and} \quad \sin\theta=\frac{e^{j\theta}-e^{-j\theta}}{2j}$$

Interpret the terms of cos and sin as pairs of points moving in the complex plane as $\theta$ increases.

**6.** In chapter 4, page 152 we made a table of values showing that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e^1=e$. In fact an alternative definition of the exponential function is $e^x=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$, where $x$ can be any complex number. Construct an Excel spreadsheet that takes $x$ as input and makes a table similar to the one on page 152. Hint: set up columns like this and let $n$ run from 1 to, say, 100:

| $n$ | $\left(1+\frac{x}{n}\right)$ | $\left(1+\frac{x}{n}\right)^n$ | $\operatorname{Re}\left(1+\frac{x}{n}\right)^n$ | $\operatorname{Im}\left(1+\frac{x}{n}\right)^n$ |
|---|---|---|---|---|

Then plot the last two columns in a scatter plot.
The figure shows the plot for two values of $x$: $x = 1$ and $x = i\pi$. We clearly see the limits.
Tip: Excel uses various character string functions to do complex arithmetic. Use these:
=complex to create a complex number,
=impower to take an integer power of it,
=imreal to take the real part,
=imaginary to take the imaginary part.



## Complex Arithmetic in Excel

Excel creates complex numbers in rectangular form using the **Complex** function. For example if cells A1 and B1 contain the real numbers 3 and 4 then typing =Complex(A1,B1) into a cell creates the complex number 3+4*i* in that cell. Typing =Complex(3,4) into the cell does the same thing. To add, subtract, multiply and divide complex numbers Excel uses the functions **imSum**, **imSub**, **imProduct** and **imDiv**. For example if cells C1 and D1 contain the complex numbers 3+4*i* and –5 +6*i* then typing =imProduct(C1,D1) into a cell multiplies them and puts the product, –39 –2*i*, into that cell. Typing =imProduct (“3+4*i*”, “–5+6*i*”) does the same thing.

Excel does not use the polar form or the exponential form of a complex number. However you can get the length and angle of a complex number by using the functions **imAbs** and **imArgument**. For example typing =imAbs(“3+4*i*”) into a cell gives the real number 5 as the magnitude or length. Typing =imArgument(“3+4*i*”) returns the real number 0.9273, which is the angle in radians. Other useful functions are **imReal** and **imaginary** to extract the real and imaginary parts of a complex number.

## Complex Arithmetic in Maple

Maple supports all three forms of a complex number: rectangular, polar and exponential. Here is how to create the rectangular number $a = 3 + 4j$, the polar number $b = 5\angle 30^\circ$, and the exponential number $c = 6e^{2j}$.

```
> a:=3+4*I; b:=polar(5,30*Pi/180); c:=6*exp(2*I);
```

$$a := 3 + 4\,\mathrm{I} \qquad b := \mathrm{polar}\left(5, \frac{1}{6}\,\pi\right) \qquad c := 6\,\mathrm{e}^{2\,\mathrm{I}}$$

Note the following:

- The imaginary unit is the capital letter *I*, and not *i* or *j*.
- polar($r$,$\theta$) means $r\angle\theta$. The angle $\theta$ must be in radians. The factor $\frac{\pi}{180}$ may be used to convert degrees to radians.
- The exponential function is exp ($I*x$), and not $e^{ix}$ or $e^{jx}$. The angle $x$ must be in radians.

To convert any form to rectangular form use the **evalc** command. Here are examples:

```
> evalc(a);evalc(b);evalc(c);
```

$$3 + 4\,\mathrm{I} \qquad \frac{5}{2}\sqrt{3} + \frac{5}{2}\,\mathrm{I} \qquad 6\cos(2) + 6\,\mathrm{I}\sin(2)$$

To get the floating point form rather than the exact form you can follow up with the **evalf** command. For example the command `evalf(evalc(c));` gives –2.496881+5.455784 I. To convert any form to polar form use the **convert(*, polar)** command. For example:

```
> convert(a,polar);convert(b,polar);convert(c,polar);
```

$$\mathrm{polar}\left(5, \arctan\left(\frac{4}{3}\right)\right) \qquad \mathrm{polar}\left(5, \frac{1}{6}\,\pi\right) \qquad \mathrm{polar}(6, 2)$$

Other useful commands are **Re** and **Im** to get the real and imaginary parts of a complex number and **abs** and **argument** to get the length and angle.

# Answers, Chapter 7 – Complex Numbers

## Problem Set 7.1 – Complex Numbers, Page 232

| | $\underline{a+b}$ | $\underline{a-b}$ | $\underline{a\cdot b}$ | $\underline{a/b}$ | $\underline{a^3}$ |
|---|---|---|---|---|---|
| **1.** | $5$ | $-1+6j$ | $15+3j$ | $-\frac{1}{6}+\frac{5}{6}j$ | $-46+9j$ |
| **2.** | $4-j$ | $-4+5j$ | $6+8j$ | $-\frac{6}{25}+\frac{8}{25}j$ | $-8j$ |
| **3.** | $7j$ | $-3j$ | $-10$ | $\frac{2}{5}$ | $-8j$ |
| **4.** | $4-6j$ | $2+2j$ | $-5-14j$ | $\frac{11}{17}+\frac{10}{17}j$ | $-9-46j$ |
| **5.** | $0$ | $-2-2j$ | $-2j$ | $-1$ | $2-2j$ |
| **6.** | $4$ | $4j$ | $8$ | $j$ | $-16+16j$ |
| **7.** | $2x$ | $2yj$ | $x^2+y^2$ | $\frac{(x^2-y^2)+(2xy)j}{x^2+y^2}$ | $\left(x^3-3xy^2\right)+$ $\left(3x^2y-y^3\right)j$ |

## Problem Set 7.2 – Complex numbers in polar coordinates, Page 234

**1.** **(a)** $3.25+j0.333 = 3.27\angle 5.86^{o}$ **(b)** $0.306-j0.252 = 0.396\angle -39.5^{o}$

**(c)** $2.98+j1.34 = 3.27\angle 24.3^{o}$ **(d)** $4.33+j13.3 = 14.0\angle 71.9^{o}$

**(e)** $9.41+j3.44 = 10.0\angle 20.1^{o}$ **(f)** $0.0423+j0.0906 = 0.100\angle 65.0^{o}$

**(g)** $10.0+j17.3 = 20.0\angle 60.0^{o}$

**2.** **(a)** $3.25$ **(b)** $13.3$

**(c)** $0.347$ **(d)** $5\cos(\omega t+45^{o})$

**(e)** $5\sin(\omega t)$ **(f)** $1\sin(\omega t-53.1^{o})$

**(g)** $0.0769\cos(12t-113^{o})$ **(h)** $0.0769e^{-5t}\cos(12t-113^{o})$

**(i)** $0.0769e^{5t}\sin(12t-67.4^{o})$

**3.** $v = 7.28\angle -44.1^{o}$

**4.** $i = 17.0\angle -80.3^{o}$

**5.** $R = 10;\quad L = 0.0173$

**6.** $R = 40;\quad L = 0.0231$

**7.** $v = 130.\angle 176^{o}$

## Problem Set 7.3 – Complex numbers in exponential form, Page 241

**1.**

| | rectangular | polar | exponential |
|---|---|---|---|
| **(a)** | $2870-j6100$ | $6740\angle 5.152$ | $6740\,e^{5.152j}$ |
| **(b)** | $29+j31$ | $42.4\angle 0.819$ | $42.4\,e^{0.819j}$ |
| **(c)** | $7+j51$ | $51.5\angle 1.43$ | $51.5\,e^{1.43j}$ |
| **(d)** | $8.29+j2.83$ | $8.76\angle 0.329$ | $8.76\,e^{0.329j}$ |
| **(e)** | $-1.405+j7.679$ | $7.81\angle 1.75$ | $7.81e^{1.75j}$ |
| **(f)** | $-41.6+j5.93$ | $42\angle 3$ | $42\,e^{3j}$ |
| **(g)** | $4.62+j.612$ | $4.66\angle .132$ | $4.66\,e^{0.132j}$ |
| **(h)** | $8.24+j15.0$ | $17.2\angle 1.07$ | $17.2e^{1.07j}$ |
| **(i)** | $-21.2-j21.2$ | $30\angle 225^{\circ}$ | $30e^{j5\pi/4}$ |

**2.**

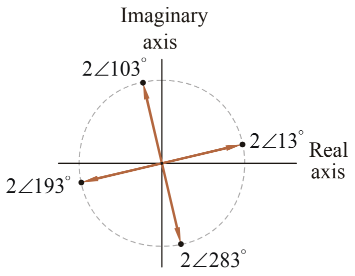
**(a)** $0.5\sin(2t-53.1^{\circ})$

**(b)** $0.0769\cos(10t-113^{\circ})$

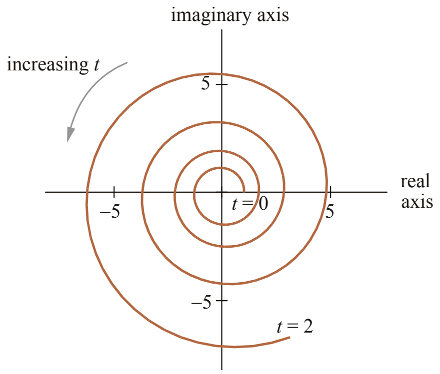
**(c)** $0.0769\,e^{-5t}\cos(12t-113^{\circ})$

**(d)** $0.0769\,e^{5t}\sin(12t-67.4^{\circ})$

**3.** Four equally spaced roots. See diagram:



**4.** Outward spiral. See diagram:



**5.** As $\theta$ increases $\frac{1}{2}e^{j\theta}$ moves counterclockwise and $\frac{1}{2}e^{-j\theta}$ moves clockwise. Multiplying (dividing) by $j$ makes it start forward (back) by $90^{\circ}$. Adding the two counter-rotating dots cancels the imaginary parts with the result that sin and cos are both real. See the diagram below.